

DPNU-96-10
 hep-ph/9603293
 March, 1996

DYNAMICAL SYMMETRY BREAKING WITH LARGE ANOMALOUS DIMENSION *

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ABSTRACT

We give an introduction to the dynamical symmetry breaking with large anomalous dimension. This is the basis of tightly bound composite Higgs models such as walking technicolor, strong ETC technicolor and top quark condensate, which are all characterized by the large anomalous dimension, $\gamma_m \simeq 1$ (walking technicolor), $1 < \gamma_m < 2$ (strong ETC technicolor) and $\gamma_m \simeq 2$ (top quark condensate) due to nontrivial short distance dynamics of the gauged Nambu-Jona-Lasinio (NJL) models (gauge theories plus four-fermion interactions).

Particular emphasis will be placed on the top quark condensate in which the critical phenomenon in the gauged NJL model yields a simple reason why the top quark can have an extremely large mass compared with other quarks and leptons. Topics will also cover a recent observation that the four-fermion theory *in the presence of gauge interactions* (gauged NJL model) can become renormalizable and nontrivial in sharp contrast to the pure NJL model without gauge interactions. The requirement of this renormalizability/nontriviality of the gauged NJL model can be applied to the top quark condensate when the standard gauge groups are unified into a GUT with “walking” coupling, which then naturally leads to the top quark and the Higgs masses both around 180 GeV.

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*To appear in Proc. 14th Symposium on Theoretical Physics “Dynamical Symmetry Breaking and Effective Field Theory”, Cheju, Korea, July 21-26, 1995

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1. Introduction

As it stands now, the standard model (SM) is a very successful framework for describing elementary particles in the low energy region, say, less than 100 GeV. However one of the most mysterious part of the theory, *the origin of mass*, has long been left unexplained. Actually, mass of *all* particles in the SM is attributed to a *single* order parameter, the vacuum expectation value (VEV) of the Higgs doublet. Thus the problem of the origin of mass is simply reduced to understanding the dynamics of the Higgs sector.

Here we note that the situation very much resembles the Ginzburg-Landau (GL)'s *macroscopic* theory for the superconductivity, the mysterious parts of which were eventually explained by the *microscopic* theory of Bardeen-Cooper-Schrieffer (BCS): The GL's phenomenological order parameter was replaced by the Cooper pair condensate due to the short range attractive forces.

A similar thing has also happened to the hadron physics where the σ model description by Gell-Mann and Levy (GML) works very well as far as the low energy (macroscopic) phenomena are concerned, while the deeper understanding of it was first given by Nambu and Jona-Lasinio (NJL)¹ based on the analogy with the BCS dynamics. Nowadays people believe that essentially the same phenomena as described by the NJL paper takes place in the microscopic theory for hadrons, QCD. In QCD the VEV of σ , the GML's order parameter $\langle \sigma \rangle = f_\pi = 93\text{MeV}$, has been replaced by the quark-antiquark pair condensate $\langle \bar{q}q \rangle = O(f_\pi^3)$, an analogue of the Cooper pair condensate, formed by the attractive color forces. The Nambu-Goldstone (NG) boson, the pion, is now a composite state of quark and antiquark. This is actually the prototype of the *dynamical symmetry breaking* (DSB) due to *composite order parameters* like fermion pair condensates.

In fact Higgs sector in the SM is precisely the same as the σ model except that $\langle \sigma \rangle = f_\pi = 93\text{MeV}$ is now replaced by the Higgs VEV= $F_\pi = 250\text{GeV}$, roughly a 2600 times scale-up. One is thus naturally led to speculate that there might exist a microscopic theory for the Higgs sector, with the Higgs VEV being replaced by the fermion-antifermion pair condensate due to yet another strong interaction called technicolor (TC)².

Unfortunately, the original version of TC was too naive to survive the FCNC (flavor-changing neutral current) syndrome³. It was not the end of the story, however. QCD-like theories (simple scale-up's of QCD) turned out *not* to be the *unique* candidate for the underlying composite dynamics of the Higgs sector.

Actually, there have been proposed a variety of composite Higgs models which, though equally behaving as the σ model in the low energy region, still have different high energy behaviors than QCD; walking technicolor^{4,5}, strong ETC technicolor^{6,7} and the top quark condensate (top mode standard model)^{8,9,10,11,12}, etc.. Interactions in these models persist at high energy or short distance and hence produce tightly bound composite Higgs.

These tightly bound composite Higgs models were actually proposed based on the explicit dynamics, gauged NJL model (gauge theory plus four-fermion interaction) within the framework of ladder Schwinger-Dyson (SD) equation. The gauged NJL model³⁷ was shown to have a phase structure divided by a critical coupling (critical line)¹⁴ similarly to the NJL model, and have a *large anomalous dimension*^{6,15,16} due to *strong attractive forces at relatively short distance or high energy*. Such a system may actually be regarded as a theory with ultraviolet fixed point(s) in contrast to the asymptotic freedom. A remarkable feature of this dynamics is that the four-fermion interaction in four dimensions may become *renormalizable* thanks to the large anomalous dimension ($\gamma_m > 1$) and the presence of gauge interaction ($\gamma_m < 2$) in sharp contrast to the pure NJL model with “ $\gamma_m = 2$ ”.^{15,16,17,18,19,20}

In this lecture we would like to give a general description of DSB with large anomalous dimension and tightly bound composite Higgs models based on that dynamics. Main parts of this subject have already been covered by many reviews^{17,21} and a textbook²². Here we shall put a stress rather on the renormalizability of the gauged NJL model and its possible application²³ to modifications of the top quark condensate in a way consistent with the recent discovery of a heavy top quark with mass about 180 GeV²⁴.

We start with basic concepts of DSB with particular emphasis on its characterization in the high energy behavior through anomalous dimension (Section 2). We then proceed to a general idea of TC and the role of anomalous dimension (Section 3). Detailed explanation will be given to how the large anomalous dimension arises in explicit dynamics, NJL model and gauged NJL model: In particular, the gauged NJL model is shown to be renormalizable in a non-perturbative sense, due to the large anomalous dimension and the very presence of gauge interaction (Section 4, 5). We demonstrate that the entire coupling space of the gauged NJL model encompasses a variety of tightly bound composite Higgs models, i.e., walking technicolor, strong ETC technicolor and top quark condensate (Section 6). In Section 7 detailed discussion will be given to the top quark condensate which actually predicted the top quark mass on the order of weak scale, exceptionally large compared with all other quarks and leptons. We give a detailed comparison between the original formulation of Miransky-Tanabashi-Yamawaki (MTY)^{8,9} and another one of Bardeen-Hill-Lindner (BHL)¹². We then discuss the renormalizability of the gauged NJL model which may be applied to a possible modification of the top quark condensate (“top mode walking GUT”).²³

2. Basic Concepts in Dynamical Symmetry Breaking

2.1. Spontaneous Symmetry Breaking

Here we first summarize basic concepts of the spontaneous symmetry breaking in a way suitable to discussions on the DSB.

2.1.1. Symmetry Realizations

Let G be a symmetry group of transformations with continuous parameters in such a way that the action $S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$ is invariant under the transformation G , $\delta\phi_i = \epsilon_A \delta^A \phi_i = -i\epsilon_A (T^A)_i^j \phi_j$, where T^A ($A = 1, 2, \dots, \dim G$) are the matrix representation of the generators of G , i.e., $\exp(-i\epsilon_A T^A) \in G$. Then there exist conserved Noether currents $J_\mu^A(x)$ corresponding to this symmetry, $\delta S = 0 \Leftrightarrow \partial^\mu J_\mu^A(x) = 0$. We may define (at least formally) conserved charge operators $Q^A \equiv \int d^3x J_0^A(x)$, $\dot{Q}^A = 0$, which, if well-defined, become generators of the symmetry group G :

$$\phi(x) \rightarrow \phi'(x) = e^{i\epsilon_A Q^A} \phi(x) e^{-i\epsilon_A Q^A}, \quad (2.1)$$

or

$$\delta^A \phi(x) = [iQ^A, \phi(x)]. \quad (2.2)$$

Let us look at n -point Green functions $G_n(x_1, x_2, \dots, x_n) \equiv \langle 0 | T \phi_1(x_1) \phi_2(x_2) \cdots \phi_n(x_n) | 0 \rangle$, where T stands for the usual time-ordered product. They have all the information of quantum field theory, i.e., field operators and the vacuum. Under the transformation of the above symmetry group G , $G_n(x_1, x_2, \dots, x_n)$ transforms as

$$G_n \rightarrow G'_n = \langle 0 | T \phi'_1(x_1) \phi'_2(x_2) \cdots \phi'_n(x_n) | 0 \rangle = \langle 0' | T \phi_1(x_1) \phi_2(x_2) \cdots \phi_n(x_n) | 0' \rangle, \quad (2.3)$$

where (2.1) and $|0'\rangle = e^{-i\epsilon_A Q^A} |0\rangle$ were used. Our principal interest lies in the variation of G_n , $\delta G_n \equiv G'_n - G_n = \epsilon_A \delta^A G_n$.

Now, there are two types of realizations (phases) of the symmetry:

(i) Wigner realization.

If a symmetry of the action is also the symmetry of the vacuum:

$$|0\rangle \rightarrow |0'\rangle = e^{-i\epsilon_A Q^A} |0\rangle = |0\rangle, \quad (Q^A |0\rangle = 0), \quad (2.4)$$

then we have $G'_n - G_n = 0$ from (2.3), i.e.,

$$\delta^A G_n = 0 \quad (2.5)$$

for *all* Green functions. Namely, $Q^A |0\rangle = 0 \implies \delta^A \forall G_n = 0$. Hence the symmetry is also the symmetry of physical states in the Fock space constructed upon this

vacuum: They are classified according to the usual representation theory of the group.

(ii) Nambu-Goldstone (NG) realization.

Conversely, if a symmetry of the action is not the symmetry of the physical states in such a way that there exists *at least one* (not necessarily all) Green function such that

$$\delta^A G_n \neq 0, \quad (2.6)$$

then the vacuum does not respect the symmetry; Symbolically,

$$|0'\rangle = e^{i\epsilon_A Q^A} |0\rangle \neq |0\rangle \quad (Q^A |0\rangle \neq 0). \quad (2.7)$$

Namely, $\delta^A \exists G_n \neq 0 \implies Q^A |0\rangle \neq 0$. We say that the symmetry is *spontaneously broken*.

The variation $\delta^A G_n$ is called an *order parameter* which in fact discriminates between the two realizations, Wigner (or disordered) phase with $\delta^A G_n = 0$ and NG (or ordered) phase with $\delta^A G_n \neq 0$. We shall call it a “*composite order parameter*” if $n \geq 2$, and an “*elementary order parameter*” if $n = 1$. Composite order parameters are in general *nonlocal* but could be local if we put $x_1 = \dots = x_n$. Local composite order parameters are often called *condensates*. From (2.3) we may write

$$\begin{aligned} \delta^A G_n(x_1, \dots, x_n) &= \langle 0 | [iQ^A, T\phi_1(x_1)\phi_2(x_2) \cdots \phi_n(x_n)] | 0 \rangle \\ &= \langle 0 | T[iQ^A, \phi_1(x_1)]\phi_2(x_2) \cdots \phi_n(x_n) | 0 \rangle \\ &\quad + \langle 0 | T\phi_1(x_1)[iQ^A, \phi_2(x_2)] \cdots \phi_n(x_n) | 0 \rangle + \cdots \\ &\quad + \langle 0 | T\phi_1(x_1)\phi_2(x_2) \cdots [iQ^A, \phi_n(x_n)] | 0 \rangle. \end{aligned} \quad (2.8)$$

Note that the commutator $[iQ^A, \phi(x)] \equiv i \int d^4 z [J_0^A(z), \phi(x)] \delta(z^0 - x^0)$ is always well-defined even when the charge Q^A itself is not.

2.1.2. Nambu-Goldstone Theorem

Now we come to the basic theorem of the spontaneous symmetry breaking, namely the NG theorem: If the continuous symmetry is spontaneously broken such that $\delta^A \exists G_n \neq 0$, then there exist massless spinless bosons (NG bosons) coupled to the currents $J_\mu^A(x)$. The proof is as follows:

Define

$$M_\mu^A(q, x_1, \dots, x_n) \equiv \int d^4 z e^{iqz} \langle 0 | T J_\mu^A(z) \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle, \quad (2.9)$$

where $\phi_1, \phi_2, \dots, \phi_n$ are fields appearing in the Lagrangian. The current conservation $\partial^\mu J_\mu^A = 0$ leads to the Ward-Takahashi (WT) identity:

$$\begin{aligned} \lim_{q_\mu \rightarrow 0} q^\mu M_\mu^A &= \lim_{q_\mu \rightarrow 0} \int d^4 z e^{iqz} (i\partial_z^\mu) \langle 0 | T J_\mu^A(z) \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle, \\ &= \langle 0 | T [iQ^A, \phi_1(x_1)] \phi_2(x_2) \cdots \phi_n(x_n) | 0 \rangle + \cdots \\ &\quad + \langle 0 | T \phi_1(x_1) \phi_2(x_2) \cdots [iQ^A, \phi_n(x_n)] | 0 \rangle \\ &= \delta^A G_n(x_1, \dots, x_n), \end{aligned} \quad (2.10)$$

where use has been made of $\partial_z^\mu T J_\mu^A(z) \phi(x) = [J_0^A, \phi(x)] \delta(z^0 - x^0)$ and (2.8). Thus, if there exists at least one Green function G_n such that $\delta^A G_n \neq 0$, then the corresponding M_μ^A must have a pole singularity at $q^2 = 0$:

$$M_\mu^A(x_1, \dots, x_n) \sim \frac{q_\mu}{q^2} \delta^A G(x_1, \dots, x_n), \quad (2.11)$$

where the Lorentz index carried by q_μ implies a spinless particle. The order parameter $\delta^A G_n \neq 0$ is nothing but a residue of the NG boson pole. This establishes existence of a massless spinless boson (NG boson) coupled to the current J_μ^A (broken current) with strength $\delta^A G_n$.

Generally, the generators of G , $\{T^A\}$, can be divided into two parts

(i) Unbroken ones $\{S^\alpha\}$ with $\delta^\alpha G_n = 0$, which span a subgroup H ($H \subset G$),

(ii) Broken ones $\{X^a\}$ with $\delta^a G_n \neq 0$: Namely,

$$\{T^A\} = \{S^\alpha \in \mathcal{H}, X^a \in \mathcal{G} - \mathcal{H}\}, \quad (2.12)$$

where \mathcal{G} and \mathcal{H} denote the algebra of G and H , respectively. In such a case G is spontaneously broken down to a subgroup H . As is obvious from the above NG theorem, there is a *one to one correspondence* between the broken current J_μ^a (broken generator X^a) and the NG boson pole in M_μ^a . Thus the number of independent NG bosons is given by that of independent broken generators, $\dim G - \dim H$.

2.2. Elementary versus Composite Order Parameters

Let us now look at the pole residue of the NG boson appearing in the current matrix $M_\mu^a(q, x_1, \dots, x_n)$:

$$M_\mu^a(q, x_1, \dots, x_n) \sim \langle 0 | J_\mu^a(0) | \pi^b(q) \rangle \frac{i}{q^2} \langle \pi^b(q) | T \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle. \quad (2.13)$$

Comparing this with (2.11), we may write the order parameter $\delta^a G_n$ as

$$\delta^a G_n(x_1, \dots, x_n) = f_\pi \cdot \langle \pi^a(q_\mu = 0) | T \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle, \quad (2.14)$$

where f_π is the “decay constant” of the NG boson defined by

$$\langle 0 | J_\mu^a(x) | \pi^b(q) \rangle = -i\delta^{ab} f_\pi q_\mu e^{-iqx}. \quad (2.15)$$

Now the n -point order parameter $\delta^a G_n$ is traded for the generic order parameter f_π multiplied by the Bethe-Salpeter (BS) amplitude $\langle \pi^a(q) | T\phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle$ which plays a role of “wave function” of the bound state π . Eq.(2.14) is another expression of WT identity or the NG theorem and is a basic formula for the soft NG boson emission vertex (low energy theorem). We may distinguish between the following two cases.

Elementary Order Parameters:

The simplest order parameter is of course “one-body” order parameter (elementary order parameter), in which case (2.14) reads:

$$\langle 0 | \delta^a \phi^b | 0 \rangle = \delta^{ab} f_\pi Z_\pi^{1/2} \quad (2.16)$$

where the elementary field ϕ becomes an interpolating field of the NG boson with the renormalization constant Z_π such that $\langle \pi^a | \phi^b | 0 \rangle = \delta^{ab} Z_\pi^{1/2}$. The linear σ model and the Higgs Lagrangian in the standard model belong to this category.

Composite order Parameters:

On the other hand, if there exists no elementary order parameters $\langle 0 | \delta^a \forall \phi^b | 0 \rangle = 0$ but $f_\pi \neq 0$, then there must exist composite order parameters $\delta^a \exists G_n \neq 0$ for $n \geq 2$ which satisfies (2.14). The NG boson π is now a composite particle with non-zero BS amplitude $\langle \pi^a(q) | T\phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle \neq 0$. In such a case there also exists a *local* composite order parameter (*condensate*) which satisfies a relation similar to (2.14):

$$\langle 0 | \delta^a (\phi_1(0) \phi_2(0) \cdots \phi_n(0)) | 0 \rangle = f_\pi \langle \pi^a | (\phi_1(0) \phi_2(0) \cdots \phi_n(0)) | 0 \rangle \neq 0. \quad (2.17)$$

This implies that the local composite operator $(\phi_1(0) \phi_2(0) \cdots \phi_n(0))$ has an overlapping with the NG boson π and becomes its interpolating field. We call this case *dynamical symmetry breaking* (DSB).^{*} QCD is a good example of this category. Note that when the elementary order parameter already exists, then usually the composite order parameters are also non-zero, while the converse is not true. Eqs.(2.14) and (2.17) are basic tools of DSB.

^{*}This definition is somewhat ambiguous when the system contains elementary scalars. Even if the VEV of scalar is zero at tree level, quantum effects may give rise to a non-zero VEV as in the Coleman-Weinberg mechanism²⁵. Similar phenomenon also takes place in the strongly coupled Yukawa model, which very much resembles the DSB in the four-fermion (NJL) model.^{27,19}

2.3. σ Model

As a well-known example having an elementary order parameter, let us recall the linear σ model. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\partial)\psi - g_{NN\pi}\bar{\psi}(\sigma + i\gamma_5\tau^a\pi^a)\psi + \frac{1}{2}\left((\partial_\mu\sigma)^2 + (\partial_\mu\pi^a)^2\right) - V(\pi, \sigma), \\ V(\pi, \sigma) &= \frac{\lambda_4}{4}\left((\sigma)^2 + (\pi^a)^2 - v^2\right)^2,\end{aligned}\quad (2.18)$$

where $\tau^a (a = 1, 2, 3)$ are Pauli matrices in isospin space of the nucleon doublet field $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$, with $g_{NN\pi}$ and λ_4 being the Yukawa coupling of the nucleon and the quartic coupling of the mesons, respectively. The meson fields $\pi^a(x)$ and $\sigma(x)$ stand for the pion (in the massless limit) and the σ meson (not existing in the Particle Data Group Table, though often regarded as a very broad resonance), respectively.

The Lagrangian can be cast into the form:

$$\begin{aligned}\mathcal{L} &= \bar{\psi}_L(i\partial)\psi_L + (L \rightarrow R) - \sqrt{2}g_{NN\pi}\left(\bar{\psi}_L M \psi_R + \bar{\psi}_R M^\dagger \psi_L\right) \\ &+ \frac{1}{2}\text{Tr}(\partial_\mu M \partial^\mu M^\dagger) + \frac{\lambda_4}{4}\left(\text{Tr}(MM^\dagger) - v^2\right)^2,\end{aligned}\quad (2.19)$$

$$\psi_{L,R} \equiv \frac{1 \mp \gamma_5}{2}\psi, \quad M \equiv \frac{1}{\sqrt{2}}(\sigma + i\pi^a\tau^a). \quad (2.20)$$

It is easy to see that this Lagrangian has a global symmetry under the $SU(2)_L \times SU(2)_R$ transformation:

$$\psi_{L,R}(x) \rightarrow \psi'_{L,R}(x) = g_{L,R}\psi_{L,R}(x), \quad (2.21)$$

$$M(x) \rightarrow M'(x) = g_L M(x) g_R^\dagger, \quad (2.22)$$

where $g_{L,R} = e^{-i\epsilon_{L,R}^a \frac{\tau^a}{2}} \in SU(2)_{L,R}$, or $\delta_{L,R}^a \psi_{L,R} = i[Q_{L,R}^a, \psi_{L,R}] = -i\frac{\tau^a}{2}\psi_{L,R}$. For $\epsilon_L^a = \epsilon_R^a = \epsilon^a$ we have a vector transformation

$$\delta^a \psi = [iQ^a, \psi] = i[Q_R^a + Q_L^a, \psi] = -i\frac{\tau^a}{2}\psi, \quad (2.23)$$

$$\delta^a M = -i\left[\frac{\tau^a}{2}, M\right] = \frac{i}{\sqrt{2}}\epsilon^{abc}\pi^b\tau^c, \quad (2.24)$$

which forms the usual isospin $SU(2)_V$ symmetry. On the other hand, for $-\epsilon_L^a = \epsilon_R^a = \epsilon_5^a$ we have an axialvector transformation

$$\delta_5^a \psi = [iQ_5^a, \psi] = i[Q_R^a - Q_L^a, \psi] = -i\gamma_5\frac{\tau^a}{2}\psi, \quad (2.25)$$

($\delta_5^a \bar{\psi} = -\bar{\psi}i\gamma_5\frac{\tau^a}{2}$). As to the axialvector transformation of M , we have

$$\delta_5^a M = i\left\{\frac{\tau^a}{2}, M\right\} = \frac{1}{\sqrt{2}}(i\delta^{ab}\tau^b\sigma - \pi^a), \quad (2.26)$$

or

$$\delta_5^a \pi^b = [iQ_5^a, \pi^b] = \delta^{ab} \sigma, \quad \delta_5^a \sigma = [iQ_5^a, \sigma] = -\pi^a. \quad (2.27)$$

Suppose that the vacuum is chosen as

$$\langle \delta_5^a \pi^b \rangle = \langle [iQ_5^a, \pi^b] \rangle = \delta^{ab} \langle \sigma \rangle \neq 0, \quad \langle \delta_5^a \sigma \rangle = \langle [iQ_5^a, \sigma] \rangle = -\langle \pi^a \rangle = 0, \quad (2.28)$$

then the chiral symmetry $SU(2)_L \times SU(2)_R$ is spontaneously broken down to $SU(2)_V$ through the *elementary order parameter*. Recalling (2.16), one finds $f_\pi Z_\pi^{1/2} = \langle \sigma \rangle$.

Actually at tree level, the wine-bottle potential in (2.18) has infinitely degenerate ground states at $\langle \sigma^2 + (\pi^a)^2 \rangle = v^2$, among which we can always choose (through appropriate $SU(2)_L \times SU(2)_R$ rotation) the unique vacuum satisfying (2.28), i.e., $f_\pi = \langle \sigma \rangle = v \neq 0$ ($Z_\pi = 1$ at level) and $\langle \pi^a \rangle = 0$. Then the spectrum of the theory can readily be read off from the Lagrangian at this vacuum. The curvature of the potential $V(\pi, \sigma)$ in (2.18) yields the (mass)²:

$$\begin{aligned} m_\sigma^2 &= \left. \frac{\partial^2 V}{\partial \sigma \partial \sigma} \right|_{\langle \sigma \rangle = v, \langle \pi \rangle = 0} = 2\lambda_4 v^2 > 0, \\ m_\pi^2 &= \left. \frac{\partial^2 V}{\partial \pi \partial \pi} \right|_{\langle \sigma \rangle = v, \langle \pi \rangle = 0} = 0, \end{aligned} \quad (2.29)$$

where the flat curvature in π^a directions corresponds to the fact that π^a are massless NG bosons.

On this spontaneous chiral symmetry breaking ($S\chi SB$) vacuum the nucleon also acquires a mass through the Yukawa coupling:

$$-g_{NN\pi} \bar{\psi} (\sigma + i\gamma_5 \tau^a \pi^a) \psi \Big|_{\langle \sigma \rangle = v, \langle \pi \rangle = 0} = -(g_{NN\pi} \langle \sigma \rangle) \bar{\psi} \psi - g_{NN\pi} \bar{\psi} (\sigma' + i\gamma_5 \tau^a \pi^a) \psi, \quad (2.30)$$

where $\sigma' \equiv \sigma - \langle \sigma \rangle$. The first term indeed behaves like a mass term, with a mass given by

$$m_N = g_{NN\pi} \langle \sigma \rangle = g_{NN\pi} v = g_{NN\pi} f_\pi \quad (2.31)$$

at tree level. This is the Goldberger-Treiman relation (with $g_A = 1$ because of tree-level). Now, it is evident that the various order parameters m_N , $\langle \sigma \rangle$, f_π are proportional to each other.

Through the above popular arguments *at Lagrangian level*, one might have gotten an impression as if the symmetry were broken already at Lagrangian (operator) level. Indeed the mass term $m_N \bar{\psi} \psi$ as it stands is not invariant under the chiral symmetry (2.21), since $\langle \sigma \rangle$ is now just a constant and no longer transform to cancel the transformation of $\bar{\psi} \psi$. However, in the case of the spontaneous symmetry breaking the symmetry is *not* broken at operator (Lagrangian) level, but solely on the vacuum. The Lagrangian must keep the invariance at any change of variables. So, what happened to the nucleon mass? Does it really break the Lagrangian symmetry? The answer is of course no.

Actually, the nucleon field ψ in (2.30) is not an appropriate field to describe this phenomenon properly at Lagrangian level. We shall explain this in terms of the nonlinear realization (For a review see Ref.26). First note that any complex matrix $M \equiv (\sigma + i\pi^a \tau^a)/\sqrt{2}$ can be written as $M = \tilde{H}U = \xi_L^\dagger H \xi_R$, where $\tilde{H}(\equiv \xi_L^\dagger H \xi_L)$ is a Hermitian matrix, and $U(\equiv \xi_L^\dagger \xi_R)$ and $\xi_L^\dagger = \xi_R = \xi$ are unitary matrices (polar decomposition). The Hermite matrix (radial mode) H can always be diagonalized as $H = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix}$ such that $\langle \hat{\sigma} \rangle = \langle \sigma \rangle = v$, while the unitary matrix (phase modes)

may be parameterized in terms of the NG bosons $\hat{\pi}$: $\xi = e^{i\hat{\pi}^a(\frac{\tau^a}{2})/v}$, where $\hat{\pi} \sim \pi$ and $\hat{\sigma} \sim \sigma$. (This polar decomposition makes sense only when $\langle H \rangle \neq 0$, i.e., the chiral symmetry is spontaneously broken.) The new fields $\xi_{L,R}$ and H transform as $\xi_{L,R} \rightarrow h \xi_{L,R} g_{L,R}^\dagger$, $H \rightarrow h H h^\dagger$, where $h \in SU(2)_V$ and $g_{L,R} \in SU(2)_{L,R}$. Then we can introduce an appropriate “nucleon” field $\Psi_{L,R} \equiv \xi_{L,R} \psi_{L,R}$, which transform as $\Psi_{L,R} \rightarrow h \Psi_{L,R}$. Now the Yukawa term reads:

$$\bar{\psi}_L M \psi_R = \bar{\Psi}_L H \Psi_R = \bar{\Psi}_L \langle H \rangle \Psi_R + \bar{\Psi}_L H' \Psi_R, \quad (2.32)$$

and ($L \leftrightarrow R$), where $H' \equiv H - \langle H \rangle$. The first term of R.H.S. in (2.32) yields precisely the same nucleon mass as before but this time it is *invariant* under the chiral symmetry, since $\Psi_{L,R}$ now transforms only under $SU(2)_V$ but not $SU(2)_L \times SU(2)_R$. Note that $\psi_{L,R} = e^{\mp i\hat{\pi}^a(\frac{\tau^a}{2})/v} \Psi_{L,R} = \Psi_{L,R} + \dots$, so that the above popular arguments at Lagrangian level are effectively still correct.

Thus the arguments at Lagrangian level are rather tricky. Here we return to rather straightforward arguments based on the Green functions which contain information on both the operator and the vacuum. Non-invariance of the Green functions are solely due to the vacuum structure but not to the operator, as we discussed in the beginning of this lecture. Let us look at (Fourier transform of) 2-point function of the nucleon, the full propagator $G_2(p) = S(p) = \mathcal{FT} \langle 0 | T(\psi(x) \bar{\psi}(0)) | 0 \rangle$, whose variation is a *composite order parameter* (see (2.8) and (2.14)):

$$\delta_5^a S(p) = \mathcal{FT} \langle 0 | T([iQ_5, \psi(x) \bar{\psi}(0)]) | 0 \rangle = f_\pi \cdot \mathcal{FT} \langle \pi^a | T(\psi(x) \bar{\psi}(0)) | 0 \rangle \neq 0. \quad (2.33)$$

From (2.25) we can rewrite the L.H.S. as

$$\delta_5^a S = S \left\{ -i\gamma_5 \frac{\tau^a}{2}, S^{-1} \right\} S = S \left(\gamma_5 \tau^a Z_\psi^{-1}(-p^2) \Sigma(-p^2) \right) S, \quad (2.34)$$

where $iS^{-1}(p) = Z_\psi^{-1}(-p^2) (\not{p} - \Sigma(-p^2))$ (in space-like region $p^2 < 0$), with $Z_\psi(-p^2)$ and $\Sigma(-p^2)$ being the wave function renormalization and mass function of the nucleon, respectively. This implies that non-invariance of the Green function or the vacuum symmetry breaking is signalled by the appearance of mass function $\Sigma(-p^2)$ of the nucleon which is massless at Lagrangian level: $\delta_5^a S(p) \neq 0 \Leftrightarrow \Sigma(-p^2) \neq 0$.

Now define the amputated BS amplitude

$$\tilde{\chi}_\pi^a(p, q) \equiv S^{-1}(p+q) \mathcal{FT} \langle \pi^a | T(\psi(x) \bar{\psi}(0)) | 0 \rangle S^{-1}(p), \quad (2.35)$$

then (2.33) reads

$$\gamma_5 \tau^a \Sigma(-p^2) = f_\pi \cdot \chi_\pi^a(p, 0) \neq 0, \quad (2.36)$$

where $\chi_\pi^a(p, 0) \equiv Z_\psi \tilde{\chi}_\pi^a(p, 0)$ is a renormalized amputated BS amplitude. At tree level we have $f_\pi = \langle \sigma \rangle$ and $\chi_\pi^a(p, 0) = \gamma_5 \tau^a g_{NN\pi} = \text{const.}$, then (2.36) implies a constant nucleon mass $\Sigma(-p^2) \equiv m_N = f_\pi g_{NN\pi} = g_{NN\pi} \langle \sigma \rangle$, which is nothing but the Goldberger-Treiman relation (2.31) obtained through the Lagrangian level arguments.

Note that the BS amplitude $\chi_\pi^a(p, 0) \neq 0$ in this case just means an already existing Yukawa vertex among the elementary fields in the Lagrangian and hence never implies that π is a “composite” of $N\bar{N}$ (at least at tree level). This reflects the fact that main origin of the symmetry breaking $f_\pi \neq 0$ is due to the elementary order parameter $\langle \delta_5^a \pi^b \rangle = \delta^{ab} \langle \sigma \rangle \neq 0$ but not due to the composite one (fermion pair condensate $\langle \bar{\psi} \psi \rangle$) related to the compositeness of π .^{*} This model is a good example to show that if there exists an elementary order parameter, then composite (nonlocal) order parameters also exist in general. However, the latter are only secondary objects in this model: For example, even when $f_\pi = \langle \sigma \rangle \neq 0$, $\Sigma(-p^2)$ could be zero (if $g_{NN\pi} = 0$).

2.4. QCD

2.4.1. Chiral Symmetry

Now we discuss QCD where no elementary order parameters exist while composite ones actually do. For simplicity we confine ourselves to the 2-flavor quarks $q = \begin{pmatrix} u \\ d \end{pmatrix}$ whose masses are very small compared with the QCD scale Λ_{QCD} , i.e., $m_u, m_d \ll \Lambda_{QCD}$. These masses, called *current masses*, are entirely due to the Higgs VEV through the Yukawa coupling in the Glashow-Salam-Weinberg model and have nothing to do with the QCD dynamics.

The QCD Lagrangian is given by

$$\mathcal{L} = \bar{q}(i\not{D} - \mathcal{M})q - \frac{1}{4}F_{\mu\nu}^\alpha F_{\alpha}^{\mu\nu} + (\text{gauge fixing}) + (\text{FP ghost}), \quad (2.37)$$

where $D_\mu = \delta^{ij}(\partial_\mu - g\frac{\lambda^\alpha}{2}A_\mu^\alpha)$ ($\alpha = 1, \dots, 8$) is a unit matrix in flavor space $i, j = (u, d)$, with g being the gauge coupling, and $F_{\mu\nu}^\alpha$ is a field strength of the gluon field A_μ^α , and the quark (current) mass matrix \mathcal{M} is diagonal with eigenvalues (m_u, m_d) . In the limit $\mathcal{M} \rightarrow 0$ the Lagrangian possesses a chiral $SU(2)_L \times SU(2)_R$ symmetry under the transformation:

$$q_{L,R}(x) \rightarrow q'_{L,R}(x) = e^{-i\epsilon_{L,R}^a \frac{\tau^a}{2}} q_{L,R}(x), \quad (2.38)$$

^{*}This statement is no longer valid if the Yukawa coupling is strong enough to trigger the symmetry breaking.^{27,19}

which is expected to be spontaneously broken down to $SU(2)_V$. The broken current is the axialvector current:

$$J_{5\mu}^a = \bar{\psi} i \gamma_5 \frac{\tau^a}{2} \psi. \quad (2.39)$$

2.4.2. Composite Order Parameters in QCD

In contrast to the σ model, QCD has no elementary order parameters. If the quark and gluon fields were order parameters, then the Lorentz invariance, color symmetry and charge symmetry would have been spontaneously broken in QCD in contrast to the reality. Then only possible order parameters are composite ones, variation of n -point Green functions or that of local composite fields. A relevant composite order parameter (2-point function) takes the same form as (2.33):

$$\delta_5^a S(p) = \mathcal{FT} \langle 0 | T ([iQ_5, q(x)\bar{q}(0)]) | 0 \rangle = f_\pi \cdot \mathcal{FT} \langle \pi^a | T (q(x)\bar{q}(0)) | 0 \rangle \neq 0, \quad (2.40)$$

which then leads to the same relation as (2.36):

$$\gamma_5 \tau^a \Sigma(-p^2) = f_\pi \cdot \chi_\pi^a(p, 0) \neq 0, \quad (2.41)$$

where $\Sigma(-p^2)$, now a *dynamical mass* of quark, signals the spontaneous chiral symmetry breaking due to the QCD dynamics. We may define an “on-shell” dynamical mass m^* as $\Sigma(m^{*2}) = m^*$, which is often called *constituent mass* (it also includes the effects of the explicit breaking due to the current mass). In contrast to the σ model where $\chi_\pi^a(p, 0) = g_{NN\pi} \tau^a \gamma_5$ (tree level), there is no Yukawa coupling $g_{qq\pi}$ at Lagrangian level in QCD. However, we have an “induced” Yukawa vertex $\chi_\pi^a(p, 0)$ which is a “wave function” of π as a composite of $q\bar{q}$ and is related to the dynamical mass $\Sigma(-p^2)$ through (2.41).

According to (2.17), we may also consider a two-body *local* composite order parameter corresponding to (2.40):

$$\langle [iQ_5^a, \bar{q} i \gamma_5 \tau^b q] \rangle = f_\pi \cdot \langle \pi^a | \bar{q} i \gamma_5 \tau^b q | 0 \rangle = \delta^{ab} \langle \bar{q} q \rangle \neq 0, \quad (2.42)$$

which implies that the composite operator $\bar{q} i \gamma_5 \tau^a q$ is an interpolating field of the composite NG boson, the pion π^a . Note that $\bar{q} i \gamma_5 \tau^a q$ and $\bar{q} q$ transform in the same way as π^a and σ in the σ model and actually are interpolating fields of the pion and the σ meson, respectively: $\pi^a \sim \bar{q} i \gamma_5 \tau^a q$, $\sigma \sim \bar{q} q$.

Thus the composite order parameters $\Sigma(-p^2)$ and $\langle \bar{q} q \rangle$ in QCD necessarily imply compositeness of the pion, $\chi_\pi^a(p, 0) \neq 0$ and $\langle \pi^a | \bar{q} i \gamma_5 \tau^a q | 0 \rangle \neq 0$, respectively.

2.4.3. High Energy Behavior of Composite Order Parameters

Now we are interested in the high energy (short distance) behavior of such composite order parameters, which can probe the underlying dynamics relevant to the composite NG boson. Actually, detailed information of the underlying dynamics is

reflected on the behavior of nonlocal order parameter $\Sigma(-p^2)$ at $-p^2 \gg \Lambda_{QCD}^2$, or equivalently that of local order parameter (condensate) $\langle 0 | (\bar{q}q)_\Lambda | 0 \rangle$ at $\Lambda \gg \Lambda_{QCD}$, where Λ is a high energy scale to renormalize the condensate. As a low energy scale we take the scale parameter of QCD, Λ_{QCD} , which is typically of order 100 MeV - 1 GeV and actually characterizes the scale of the order parameters $f_\pi \simeq 93\text{MeV}$, $m^* \simeq 300\text{MeV}$ or $\langle \bar{q}q \rangle_{\Lambda_{QCD}} \simeq -(250\text{MeV})^3$.

We first study a local composite order parameter, a condensate $\langle \bar{\psi}\psi \rangle$ for a generic fermion ψ . The renormalization of the condensate operator is

$$(\bar{\psi}\psi)_0 \equiv (\bar{\psi}\psi)_\Lambda = Z_m^{-1} (\bar{\psi}\psi)_\mu, \quad (2.43)$$

where the suffix 0 stands for a bare quantity at UV cutoff Λ ($\gg \mu$, μ : reference renormalization point) and $Z_m = Z_m(\frac{\Lambda}{\mu})$ is the renormalization constant of the current quark mass

$$m_0 \equiv m_\Lambda = Z_m m_\mu, \quad (2.44)$$

so that the mass term is renormalized as $m_0(\bar{\psi}\psi)_0 = m_\mu(\bar{\psi}\psi)_\mu$. We introduce anomalous dimension

$$\gamma_m(g_\mu) \equiv \mu \frac{\partial}{\partial \mu} \ln Z_m. \quad (2.45)$$

This can be inverted into

$$Z_m = \exp \left[- \int_0^{t_\Lambda} \gamma_m(t') dt' \right], \quad (2.46)$$

where we have defined $\gamma_m(t) \equiv \gamma_m(\bar{g}(t))$ in terms of a running coupling $\bar{g}(t)$ which satisfies $\frac{d}{dt} \bar{g}(t) = \beta(\bar{g}(t))$ such that $\bar{g}(0) = g_\mu$, with $\beta(g_\mu) \equiv \mu \frac{\partial}{\partial \mu} g_\mu$ and $t \equiv \frac{1}{2} \ln \frac{-p^2}{\mu^2}$ ($t_\Lambda \equiv \frac{1}{2} \ln \frac{\Lambda^2}{\mu^2}$). Thus the renormalization effects on the condensate at high energy scale $\Lambda (\gg \mu = O(\Lambda_{QCD}))$ is governed by the anomalous dimension:

$$\langle \bar{\psi}\psi \rangle_\Lambda = Z_m^{-1} \langle \bar{\psi}\psi \rangle_\mu = \exp \left[\int_0^{t_\Lambda} \gamma_m(t') dt' \right] \cdot \langle \bar{\psi}\psi \rangle_\mu. \quad (2.47)$$

The positive anomalous dimension enhances the condensate at high energy scale. The above result remains the same in the chiral symmetry limit $m_\mu \rightarrow 0$.

The same enhancement factor due to the anomalous dimension also appears in the nonlocal order parameter. Using the Wilson's operator product expansion (OPE), we expand the product of operators in terms of a series of local composite operators at short distances:

$$T \left(\psi(x) \bar{\psi}(0) \right) \stackrel{x \rightarrow 0}{\sim} C_1(x) \mathbf{1} + C_{\bar{\psi}\psi}(x) (\bar{\psi}\psi)_\mu + \dots, \quad (2.48)$$

where each term is factorized into the x -dependent c-numbers $C_i(x)$ called Wilson coefficients and the x -independent renormalized local composite operators O_i . The operators are placed in the order of increasing canonical mass dimension; $\mathbf{1}$ is a

unit operator (dimension 0), $(\bar{\psi}\psi)_\mu$ is a quark condensate operator (dimension 3) renormalized at μ and \dots stands for the operators of higher dimensions. Since the L.H.S. has a definite mass dimension ($= 3$ in this case), the coefficient corresponding to lower dimensional operator is expected to be more singular (more dominant) at $x \rightarrow 0$: $C_i(x) \sim x^{-3+d_i}$ for the operator with dimension d_i . Taking Fourier transform of (2.48), we have

$$-iS(p) \stackrel{p \rightarrow \infty}{\sim} C_1(p)\langle 1 \rangle + C_{\bar{\psi}\psi}(p)\langle \bar{\psi}\psi \rangle_\mu + \dots, \quad (2.49)$$

which should be compared with the expansion of L.H.S.:

$$-iS(p) = Z_\psi(-p^2) \frac{1}{\not{p} - \Sigma(-p^2)} \stackrel{p \rightarrow \infty}{\simeq} Z_\psi(-p^2) \left(\frac{1}{\not{p}} + \frac{\Sigma(-p^2)}{p^2} + \dots \right), \quad (2.50)$$

where $Z_\psi(-p^2)$ is corresponding to a wave function renormalization of the fermion field and $Z_\psi(-p^2) \rightarrow 1$ ($-p^2 \rightarrow \infty$) in Landau gauge. (Hereafter we confine ourselves to the Landau gauge.) If the corresponding operators were scaling according to the canonical dimension, then $C_1(p)$ and $C_{\bar{\psi}\psi}(p)$ would behave as $C_1(p) \sim 1/\not{p}$ and $C_{\bar{\psi}\psi}(p) \sim 1/(p^2)^2$, respectively: Namely,

$$\Sigma(-p^2) \sim \frac{\langle \bar{\psi}\psi \rangle_\mu}{p^2}. \quad (2.51)$$

However, the behavior of C_1 and $C_{\bar{\psi}\psi}$ will be modified by the renormalization effects through the anomalous dimension. The Wilson coefficients satisfy the renormalization-group equation (RGE) whose solutions read:

$$\begin{aligned} C_1(p) &= \exp \left[- \int_0^t 2\gamma_\psi(t') dt' \right] \frac{1}{\not{p}} + \dots, \\ C_{\bar{\psi}\psi}(p) &= c_{\bar{\psi}\psi}(p) \frac{1}{(p^2)^2} \langle \bar{\psi}\psi \rangle_\mu \exp \left[\int_0^t \gamma_m(t') - 2\gamma_\psi(t') dt' \right] + \dots, \end{aligned} \quad (2.52)$$

where $\gamma_\psi \equiv \mu \frac{\partial}{\partial \mu} \ln Z_\psi^{1/2}$, with $Z_\psi = Z_\psi(\Lambda/\mu)$ being the fermion wave function renormalization constant, and $c_{\bar{\psi}\psi}(p)$ is a part of the Wilson coefficients not determined by RGE alone (determined by details of the dynamics). Comparing both sides of (2.49), i.e., (2.50) and (2.52), we get

$$Z_\psi(-p^2) \stackrel{-p^2 \rightarrow \infty}{\simeq} \exp \left[- \int_0^t 2\gamma_\psi(t') dt' \right], \quad (2.53)$$

$$\Sigma(-p^2) \stackrel{-p^2 \rightarrow \infty}{\simeq} \frac{\langle \bar{\psi}\psi \rangle_\mu}{p^2} \cdot c_{\bar{\psi}\psi}(p) \exp \left[\int_0^t \gamma_m(t') dt' \right]. \quad (2.54)$$

Note that order parameters are in general proportional to each other: The nonlocal order parameter $\Sigma(-p^2)$ is in fact proportional to the local one $\langle \bar{\psi}\psi \rangle_\mu$. Moreover, the nonlocal order parameter $\Sigma(-p^2)$ has the same enhancement factor

$$\exp \left[\int_0^t \gamma_m(t') dt' \right] \quad (2.55)$$

as the local one $\langle \bar{q}q \rangle_\Lambda$ in (2.47), with a simple replacement $t_\Lambda \rightarrow t$.

2.4.4. High Energy Behavior of Composite Order Parameters in QCD

Now, the QCD coupling is asymptotically free and small in the high energy region. The one-loop beta function and the anomalous dimension are given by $\beta(g) = -bg^3$ with $b = \frac{1}{(4\pi)^2}(11N_c - 2N_f)/3$, and $\gamma_m = cg^2$ where $c = \frac{1}{(4\pi)^2}6C_2(\mathbf{F})$, with $C_2(\mathbf{F}) = (N_c^2 - 1)/(2N_c)$ being the quadratic Casimir of the fermion representation $\mathbf{F} = \mathbf{N}_c$ ($N_c = 3$). Then the anomalous dimension is logarithmically vanishing:

$$\gamma_m(\bar{g}(t)) = c\bar{g}(t)^2 = \frac{A}{2\bar{t}} \quad (\bar{t} \equiv \frac{1}{2} \ln \frac{-p^2}{\Lambda_{QCD}^2} = t + \ln \frac{\mu}{\Lambda_{QCD}}), \quad (2.56)$$

with $A = c/b = 24/(33 - 2N_f)$. This gives rise to (only) a logarithmic enhancement of the canonical result:

$$\exp \left[\int_0^t \gamma_m(t') dt' \right] \simeq \exp \left[\frac{A}{2} \ln \left(\frac{\bar{t}}{\bar{t}_\mu} \right) \right] = \left(\frac{\ln \frac{-p^2}{\Lambda_{QCD}^2}}{\ln \frac{\mu^2}{\Lambda_{QCD}^2}} \right)^{\frac{A}{2}}. \quad (2.57)$$

More specifically, (2.47) reads

$$\langle \bar{q}q \rangle_\Lambda = \left(\frac{\ln \frac{\Lambda}{\Lambda_{QCD}}}{\ln \frac{\mu}{\Lambda_{QCD}}} \right)^{\frac{A}{2}} \langle \bar{q}q \rangle_\mu, \quad (2.58)$$

and (2.54) reads²⁸

$$\Sigma(-p^2) \stackrel{-p^2 \rightarrow \infty}{\simeq} \frac{1}{p^2} \left[\frac{\frac{2\pi^2 A}{N_c}}{\left(\ln \frac{\mu^2}{\Lambda_{QCD}^2} \right)^{\frac{A}{2}}} \cdot \langle \bar{q}q \rangle_\mu \right] \cdot \left(\ln \frac{-p^2}{\Lambda_{QCD}^2} \right)^{\frac{A}{2}-1}, \quad (2.59)$$

where the extra $(\log)^{-1}$ factor came from $c_{\bar{\psi}\psi}(-p^2) = (1/4N_c)3C_2(\mathbf{F})\bar{g}^2 = (\pi^2 A)/(N_c \bar{t})$, which is due to one-gluon exchange graph in Landau gauge, the lowest diagram giving rise to the condensate operator in OPE.^{28,22*} Thus the dynamical mass is rapidly damping in high energy, $1/p^2$, roughly the result of canonical dimension arguments up to logarithm. Since the QCD coupling is vanishingly small in high energy, quantum corrections due to vanishingly small anomalous dimension are accordingly very small, only logarithmic deviation. This implies that $S\chi SB$ effects diminish rapidly when the coupling (attractive force) tends to zero in high energy.

*Only with this extra log factor, the mass function can correctly reproduce (2.58) through the condensate integral;

$$\langle \bar{q}q \rangle = -Tr S(p) = -\frac{N_c}{4\pi^2} \int_0^{\Lambda^2} dx \frac{x\Sigma(x)}{x + \Sigma(x)^2}, \quad x \equiv -p^2 > 0. \quad (2.60)$$

2.4.5. Pagels-Stokar Formula for f_π

The generic order parameter f_π defined by (2.15) for the axialvector current (2.39) is the decay constant of the pion which may be calculated through graphical consideration:

$$f_\pi q_\mu \delta^{ab} = i \langle 0 | J_{5\mu}^a(0) | \pi^b(q) \rangle = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[S(p+q) \chi_\pi^b(p, q) S(p) \gamma_\mu \gamma_5 \frac{\tau^a}{2} \right], \quad (2.61)$$

where the amputated BS amplitude of the NG boson $\chi_\pi^a(p, q)$ may be determined by the BS equation. Instead of solving the BS equation, however, here we use the famous Pagels-Stokar (PS) formula²⁹ which expresses the decay constants in terms of dynamical mass function $\Sigma(-p^2)$ of the condensed fermion. Taking derivative of (2.61) with respect to q^ν and setting $q^\nu = 0$, we get

$$g_{\mu\nu} \delta^{ab} f_\pi = \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\frac{\partial S(p)}{\partial p^\nu} \chi^b(p, 0) S(p) \gamma_\mu \gamma_5 \frac{\tau^a}{2} + S(p) \chi_\pi^b(p, 0)' S(p) \gamma_\mu \gamma_5 \frac{\tau^a}{2} \right], \quad (2.62)$$

where $\chi_\pi^a(p, q)' \equiv (\partial/\partial p^\nu) \chi_\pi^a(p, q)$.

The PS formula is obtained simply by ignoring the second term with the derivative χ_π' . This is known to be a good approximation when the ladder approximation is good.⁴⁴ Then we only have to evaluate the first term which is written *only in terms of the mass function* $\Sigma(-p^2)$, since $\chi_\pi(p, 0)$ is related to $\Sigma(-p^2)$ through the WT identity (2.41):

$$\chi_\pi^a(p, 0) = \frac{\gamma_5 \tau^a \Sigma(-p^2)}{f_\pi}. \quad (2.63)$$

Thus we obtain the PS formula for the NG boson in the spontaneous symmetry breaking $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$:

$$f_\pi^2 = \frac{N_c}{4\pi^2} \int_0^\infty dx \cdot x \frac{\Sigma(x)^2 - \frac{x}{4} \frac{d}{dx} \Sigma^2(x)}{(x + \Sigma^2(x))^2}, \quad (2.64)$$

where $x \equiv -p^2$.

We can easily see that f_π also depends on the anomalous dimension or the damping rate of the mass function, though its dependence is much milder than that of the condensate. Although the rapid damping mass function (2.59) trivially yields convergent integral for f_π in the QCD case, its convergence is highly nontrivial in the general case and is actually related to the renormalizability/nontriviality of the gauged NJL model whose mass function is very slowly damping due to large anomalous dimension as will be discussed in later sections.

2.5. High Energy Theories and Anomalous Dimension

If in contrast to QCD the theory has a non-vanishing anomalous dimension $\gamma_m(t) \simeq \gamma_m \neq 0$ due to non-vanishing coupling constant (behaving as a nontrivial ultraviolet

(UV) fixed point/pseudo fixed point) at high energies, then we have a *power enhancement* instead of the logarithmic one in QCD:

$$\exp \left[\int_0^t \gamma_m(t') dt' \right] \simeq e^{\gamma_m t} = \left(\frac{-p^2}{\mu^2} \right)^{\gamma_m/2}. \quad (2.65)$$

Accordingly, we have power-enhanced order parameters for the generic fermion ψ :

$$\Sigma(p^2) \simeq \frac{\langle \bar{\psi}\psi \rangle_\mu}{p^2} \cdot \left(\frac{-p^2}{\mu^2} \right)^{\gamma_m/2}, \quad (2.66)$$

$$\langle \bar{\psi}\psi \rangle_\Lambda = Z_m^{-1} \langle \bar{\psi}\psi \rangle_\mu \simeq \left(\frac{\Lambda}{\mu} \right)^{\gamma_m} \cdot \langle \bar{\psi}\psi \rangle_\mu. \quad (2.67)$$

This is actually the mechanism that Holdom³⁰ proposed *without explicit dynamics* to resolve the problems of FCNC and the light pseudo NG bosons in technicolor, by simply assuming $\gamma_m \geq 1$.

3. Technicolor

3.1. Scaling-Up QCD

Technicolor (TC)³ is replacing the elementary Higgs fields in the SM by composite mesons, analogues of π and σ in QCD, which are made out of hypothetical fermions called technifermions interacting via hypothetical gauge interactions with the gauge bosons called technigluons. Actually, the Higgs Lagrangian in SM

$$\mathcal{L}_{\text{Higgs}} = |\partial_\mu \phi|^2 - \lambda_4 \left(|\phi|^2 - \frac{1}{2}v^2 \right)^2 \quad (3.1)$$

is precisely the same as the bosonic part of the σ model in (2.18), when we rewrite ϕ as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} i\pi_1 + \pi_2 \\ \sigma - i\pi_3 \end{pmatrix}. \quad (3.2)$$

The transformation property can easily be checked through the identification of 2×2 matrix M in (2.20) with $M = (\tilde{\phi}, \phi)$, where $\tilde{\phi} = i\tau_2 \phi^*$: $\phi \rightarrow g_L \phi$, $\tilde{\phi} \rightarrow g_L \tilde{\phi}$. The only difference is the scale of the order parameters: $v = F_\pi = 250\text{GeV}$ in the Higgs Lagrangian in contrast to $v = f_\pi = 93\text{MeV}$ in the σ model for hadrons, roughly 2600 times larger.

Then the simplest idea to regard the Higgs fields ϕ as composites would be to scale-up QCD: $\Lambda_{QCD} \rightarrow \Lambda_{TC} \simeq 2600\Lambda_{QCD}$. Thus the low energy limit of the TC is precisely described by the Higgs Lagrangian (3.1). When we switch on the $SU(2)_L \times U(1)_Y$ gauge interactions to the technifermion doublet ψ , the composite NG bosons $\pi^a \sim i\psi\gamma_5\tau^a\psi$ are absorbed into W and Z bosons through dynamical Higgs mechanism³:

$$m_W^2 = \left(\frac{g_2}{2} F_{\pi^\pm} \right)^2, \quad m_Z^2 \cos^2 \theta_W = \left(\frac{g_2}{2} F_{\pi^0} \right)^2, \quad (3.3)$$

where g_2 is the $SU(2)_L$ gauge coupling. $F_{\pi^\pm}, F_{\pi^0} \simeq 250\text{GeV}$ are the decay constants of the composite NG bosons π^\pm, π^0 to be absorbed into W and Z bosons, respectively, and determine the scale of technifermion dynamical mass, or the technifermion condensate.

3.2. Need for Large Anomalous Dimension

This is a beautiful idea to account for the origin of W, Z boson masses. What about the mass of quarks and leptons, then? We would need some interactions to communicate the composite Higgs sector (technifermion condensate) to the quarks and leptons, analogues of the Yukawa interactions in σ model and SM itself. Extended TC (ETC)³ would be the simplest idea to give rise to such (effective four-fermion) interactions among technifermions and quarks/leptons through the ETC gauge symmetry. ETC unifies the technifermions and quarks/leptons into the same multiplets and then split them in the course of spontaneous symmetry breaking of the ETC gauge symmetry at somewhat higher scales called ETC scales $\Lambda_{ETC} \gg \Lambda_{TC}$. The effective four-fermion couplings between the quark/lepton mass operators and the technifermion condensates $\langle \bar{\psi}\psi \rangle$ are characterized by the ETC scales as $1/\Lambda_{ETC}^2$. Then the quarks/leptons masses are given by

$$m \simeq \frac{-\langle \bar{\psi}\psi \rangle_{\Lambda_{ETC}}}{\Lambda_{ETC}^2}. \quad (3.4)$$

If we were assuming QCD-like theories, then (2.58) would imply $\langle \bar{\psi}\psi \rangle_{\Lambda_{ETC}} \simeq \langle \bar{\psi}\psi \rangle_{\Lambda_{TC}}$ up to small logarithmic corrections. This is disastrous, since ETC unification also produces FCNC with strength of the same order, e.g., $\sim (\bar{s}_L \gamma_\mu d_L)^2 / \Lambda_{ETC}^2$, which is, however, bounded by the experiments of $K^0 - \bar{K}^0$ mixing as $1/\Lambda_{ETC}^2 < 10^{-5} \text{TeV}^{-2}$ in the case of s quark. Since $\langle \bar{\psi}\psi \rangle_{\Lambda_{TC}} \simeq -(\Lambda_{TC})^3 \simeq -(250\text{GeV})^3$, or $\Lambda_{TC}/\Lambda_{ETC} < 10^{-3}$ in the typical TC model, (3.4) implies at most $m_s \simeq 0.1\text{MeV}$, i.e., 10^{-3} smaller than the reality. Thus the TC as a naive QCD scale-up was dead in the early 80's.³

However, the situation is drastically changed in the theory with large anomalous dimension $\gamma_m \geq 1$.³⁰ Through (2.67) such a theory will actually yield a large enhancement factor 10^3 as desired:

$$\frac{\langle \bar{\psi}\psi \rangle_{\Lambda_{ETC}}}{\langle \bar{\psi}\psi \rangle_{\Lambda_{TC}}} \simeq \left(\frac{\Lambda_{ETC}}{\Lambda_{TC}} \right)^{\gamma_m} \geq \frac{\Lambda_{ETC}}{\Lambda_{TC}} > 10^3. \quad (3.5)$$

There is another syndrome of TC (not ETC), namely, the typical TC models (“one-family model”) predict many pseudo NG bosons (technipions) in several GeV region, which are already ruled out by experiments.³ However, the same enhancement factor due to anomalous dimension simultaneously resolves this problem by raising their masses to $O(\Lambda_{TC})$:³⁰

$$m_{\text{pNG}}^2 \sim \frac{(\langle \bar{\psi}\psi \rangle_{\Lambda_{ETC}})^2}{F_\pi^2 \Lambda_{ETC}^2} \geq \frac{(\langle \bar{\psi}\psi \rangle_{\Lambda_{TC}})^2}{\Lambda_{TC}^4} \simeq \Lambda_{TC}^2, \quad (3.6)$$

where $F_\pi \simeq \Lambda_{TC}$ was granted. Thus the large anomalous dimension amplifies a small symmetry violation (explicit chiral symmetry breaking, etc.) to the full strength through the enhancement factor in the condensate.

3.3. Other Aspects of Large Anomalous Dimension

Such a large enhancement also amplifies the small symmetry violation of high energy parameters not only in the condensate but in the mass function itself particularly for the case $\gamma_m > 1$ ($\neq 1$). This fact was first utilized by MTY^{8,9,17} based on an explicit dynamics in the proposal of a top quark condensate ($m_t \gg m_{b,c,\dots}$), and was later re-emphasized in a slightly different context^{31,32}.

Large anomalous dimension actually implies a tightly bound NG bosons due to relatively short distance dynamics. In fact, from (2.41) and (2.66) we have the high energy behavior of the amputated BS amplitude of the NG bosons at zero NG-boson-momentum:

$$\chi_\pi^a(p, 0) = \frac{1}{F_\pi} \gamma_5 \tau^a \Sigma(-p^2) \sim \left(\frac{-p^2}{\mu^2} \right)^{-1 + \frac{1}{2}\gamma_m}. \quad (3.7)$$

In QCD with $\gamma_m \simeq 0$ we find $\chi_\pi \sim (-p^2/\mu^2)^{-1}$ and hence the radius of the interaction within the composite $\langle r \rangle \simeq \mu^{-1} \simeq F_\pi^{-1}$, in which case the σ model description obviously breaks down at the order of $O(F_\pi)$. On the other hand, in the extreme case of $\gamma_m \simeq 2$ we have $\chi_\pi \sim \text{const.}$ and hence $\langle r \rangle \simeq \Lambda^{-1}$ (almost point-like interaction range, or very tightly bound), in which case the σ model description persists up to the high energy scale Λ .

4. Nambu-Jona-Lasinio Model

Do such explicit dynamical models as have large anomalous dimension really exist? The answer is yes. We shall explain solutions of ladder SD equation for the gauged NJL models (gauge theories plus four-fermion theories) which actually have large anomalous dimension $1 < \gamma_m < 2$. They encompass a variety of tightly bound composite Higgs models, such as walking TC ($\gamma_m \simeq 1$), strong-ETC TC ($1 < \gamma_m < 2$) and top quark condensate ($\gamma_m \simeq 2$), etc..

Here we start with discussion on the NJL model¹. The NJL model is of course non-renormalizable and trivial theory, i.e., we cannot take the UV cutoff to infinity to have a sensible continuum theory, in contrast to the gauged NJL model. Nevertheless it has important common features with the gauged NJL model and is a pedagogical tool to understand physics of the large anomalous dimension.

4.1. Gap Equation

Let us consider an NJL model invariant under chiral $SU(2)_L \times SU(2)_R$ transformation:

$$\mathcal{L}_{\text{NJL}} = \bar{\psi} i \not{\partial} \psi + \frac{1}{2N} G \left[(\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \tau^a \psi)^2 \right], \quad (4.1)$$

where G is a four-fermion coupling and ψ is an N -component fermion. In the large N limit, we have a famous gap equation corresponding to the Hartree-Fock self-consistent equation for the fermion dynamical mass $\Sigma(-p^2) = m^* = \text{const.}$: $(1/2N)G (\bar{\psi} \psi)^2 \Rightarrow (G/N) \langle \bar{\psi} \psi \rangle \bar{\psi} \psi = -m^* \bar{\psi} \psi$. (Hereafter we will use m instead of m^* for the dynamical mass; $m^* \Rightarrow m$.) This yields a famous gap equation¹:

$$\begin{aligned} m &= -\frac{G}{N} \langle \bar{\psi} \psi \rangle = \frac{G}{N} \text{Tr} S(p) = 4G \int \frac{d^4 p}{(2\pi)^4 i} \frac{m}{m^2 - p^2} \\ &= m \cdot \frac{G}{4\pi^2} \left(\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right), \end{aligned} \quad (4.2)$$

where we introduced a UV cutoff Λ for the Euclidean momentum $p^2 < \Lambda^2$ (Hereafter we will use Euclidean momentum $-p^2 \Rightarrow p^2 (> 0)$).

There are two solutions to (4.2):

- (i) $m = 0$ (unbroken solution),
- (ii) $m \neq 0$ such that $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$ ($S_\chi SB$ solution), in which case (4.2) can be rewritten as

$$\ln \frac{\Lambda^2}{m^2} \cdot \left(\frac{m}{\Lambda} \right)^2 \simeq \frac{1}{g^*} - \frac{1}{g}, \quad (4.3)$$

where $g \equiv G\Lambda^2/(4\pi^2)$ and $g^* = 1$. We shall refer to this relation as a scaling relation hereafter. While the first solution always exist for any coupling, the second one does only for strong coupling $g \equiv \frac{GN\Lambda^2}{4\pi^2} > g^* = 1$ for which the second solution is more stable than the first one. Thus the system is in Wigner phase (i) at $g < g^*$ while in NG phase (ii) at $g > g^*$. Both of the phases are connected to each other continuously at the critical point $g = g^* = 1$: Namely, a second order phase transition takes place. Near the vicinity of the critical coupling $g^* = 1$, we may ignore the logarithmic factor in (4.3):

$$\left(\frac{m}{\Lambda} \right)^2 \simeq \frac{1}{g^*} - \frac{1}{g}. \quad (4.4)$$

First we note that the mass rises sharply from zero to $O(\Lambda)$ as we move the coupling from $g < g^* = 1$ to $g > g^*$. This implies that if different fermions have different g 's, with $g > g^*$ for some and $g < g^*$ for others, although on the same order of $O(1)$, then the former fermions acquire large mass while the latter ones remain massless. Thus the small asymmetry among couplings results in large difference in fermion masses. This *amplification of symmetry violation* is a salient feature of the

critical phenomenon and was first used by MTY^{8,9} in the proposal of the top quark condensate.

There are two ways to look at the scaling relation (4.4) in approaching the critical point $g = g^*$: Since the dimensionless coupling g is a function of only the ratio Λ/m , we may regard $\Lambda/m \rightarrow \infty$ as either the limit of $m \rightarrow 0$ with $\Lambda = \text{fixed}$ or that of $\Lambda \rightarrow \infty$ with $m = \text{fixed}$. In the first picture we stay in the cutoff theory and requires *fine-tuning* of the coupling $1/g^* - 1/g \ll 1$ in order to realize the hierarchy $m \ll \Lambda$.

4.2. Continuum Limit, or Renormalization

In the second picture, on the other hand, we take the continuum limit $\Lambda \rightarrow \infty$ according to the RGE point of view. The coupling is now required to get renormalized and runs depending on the cutoff Λ , with the beta function being calculated from (4.4):

$$\beta(g) \equiv \Lambda \frac{\partial}{\partial \Lambda} g(\Lambda) = 2g \left(1 - \frac{g}{g^*} \right). \quad (4.5)$$

This implies existence of a nontrivial UV fixed point at the critical point: $g(\Lambda) \rightarrow g^*$ as $\Lambda \rightarrow \infty$. From (4.2) the condensate reads:

$$\langle \bar{\psi}\psi \rangle_\Lambda = -\text{Tr} S(p) \simeq -\frac{mN}{4\pi^2} \Lambda^2 \simeq \left(\frac{\Lambda}{m} \right)^2 \cdot \langle \bar{\psi}\psi \rangle_m \equiv Z_m^{-1} \langle \bar{\psi}\psi \rangle_m \quad (4.6)$$

up to logarithm. Then we have a large anomalous dimension at the UV fixed point:³³

$$\gamma_m(g = g^*) \equiv -\Lambda \frac{\partial}{\partial \Lambda} \ln Z_m = 2. \quad (4.7)$$

Thus the “fine-tuning” has been traded for the RGE concepts, namely, a nontrivial UV fixed point and large anomalous dimension.

It should be mentioned that the above RGE arguments are only *formal*, since the NJL model is well known to be a *nonrenormalizable* and *trivial* theory. The above renormalization procedure only removed the quadratic divergence, while the *logarithmic* one in (4.3) is actually a trouble for the renormalization. As we explicitly do in the later section, we may write the NJL model into the form of the Yukawa model, consisting of the original fermion and the auxiliary scalar field ϕ which have no kinetic term $(\partial_\mu \phi)^2$ and no quartic coupling $\lambda_4 \phi^4$. These terms are actually induced by the fermion loop through the Yukawa coupling and are logarithmically divergent $\sim \ln \Lambda$ as in the case of the ordinary Yukawa model. In contrast to the Yukawa model, however, the NJL model has *no counter terms* like $(\bar{\psi}\psi)^4$ (quartic coupling of ϕ) and $\partial_\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)$ (kinetic term of ϕ) in the original Lagrangian and hence these logarithmic divergences cannot be renormalized. Thus the NJL model is *not renormalizable*. We may rescale the induced kinetic term of ϕ into the usual one by rescaling ϕ , then the logarithmic divergence moves over to the rescaled Yukawa coupling (effective Yukawa coupling) which now behaves as $(1/\ln \Lambda)^{1/2} \rightarrow 0$ as $\Lambda \rightarrow$

∞ . This can also be seen from the PS formula for f_π^2 (exact at $1/N$ leading order in the NJL model), (2.64), which is logarithmically divergent since $\Sigma(p^2) \equiv m$. Accordingly, the effective Yukawa coupling $y \equiv \sqrt{2}m/f_\pi$ vanishes as the above. This is nothing but a *triviality* of the NJL model. Then the above UV fixed point is a Gaussian fixed point (free theory) and does not produce a sensible interacting continuum theory.

4.3. Renormalization in $D(2 < D < 4)$ Dimensions

Nevertheless, it was shown^{33,34,16} that the above characteristic features of the NJL model, the UV fixed point and large anomalous dimension, become true for the NJL model in $D(2 < D < 4)$ dimensions (NJL_{<4}) which are known to be *renormalizable* and *nontrivial* in the continuum limit $\Lambda \rightarrow \infty$.³⁵ The fine-tuning of NJL_{<4} model is in fact connected with a sensible continuum theory, in sharp contrast to the NJL model.

Actually, in D dimensions the scaling relation (4.3) and the condensate (4.6) become¹⁶

$$\frac{2}{4-D} \cdot \left(\frac{m}{\Lambda}\right)^{D-2} \simeq \frac{1}{g^*} - \frac{1}{g}, \quad (4.8)$$

$$\langle \bar{\psi}\psi \rangle_\Lambda \simeq \left(\frac{\Lambda}{m}\right)^{D-2} \cdot \langle \bar{\psi}\psi \rangle_m \quad (4.9)$$

at $1/N$ leading order, where $g^* = D/2 - 1$. Similarly to (4.5) and (4.7), RGE functions can be obtained from (4.8) and (4.9):

$$\begin{aligned} \beta(g) &= (D-2)g \left(1 - \frac{g}{g^*}\right), \\ \gamma_m &= D-2. \end{aligned} \quad (4.10)$$

The factor $1/(2 - D/2)$ in L.H.S. of (4.8) reflects the logarithmic divergence of (4.3) in the limit $D \rightarrow 4$. This *absence of the logarithmic divergence* in $D(2 < D < 4)$ dimensions in contrast to $D = 4$ is also true for the induced kinetic term and the induced quartic coupling of ϕ in NJL_{<4} model, which are indeed *finite* at $\Lambda \rightarrow \infty$ due to the $D(< 4)$ dimensional momentum integral.

Then we can perform explicit renormalization of the NJL_{<4} model by simply removing power divergences in the effective potential.^{34,16} This leads to the RGE functions, $\beta(g)$ and $\gamma_m(g)$, written in terms of the *renormalized coupling* g in the *continuum* theory ($\Lambda \rightarrow \infty$):^{33,34,16}

$$\begin{aligned} \beta(g) &= (D-2)g \left(1 - \frac{g}{g^*}\right), \\ \gamma_m(g) &= (D-2)\frac{g}{g^*}, \end{aligned} \quad (4.11)$$

where $\beta \equiv (\mu\partial/\partial\mu)g$ and $\gamma_m \equiv (\mu\partial/\partial\mu)\ln Z_\mu$, with μ being the renormalization point. The above expressions are obtained from effective potential and hence valid

both in the unbroken and $S\chi SB$ phases. These RGE functions take the same form in the bare coupling, which coincide with (4.10) obtained from the gap equation (scaling relation) and the condensate at $g \simeq g^*$ in the $S\chi SB$ phase.

In terms of the anomalous dimension, we may regard the renormalizability of the $\text{NJL}_{<4}$ model as follows: The fact that we can renormalize the theory without higher dimensional operators $(\bar{\psi}\psi)^4$ (quartic coupling of ϕ) and $\partial_\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)$ (kinetic term of ϕ) at $1/N$ leading order simply reflects the following fact: $(\bar{\psi}\psi)^2$ is a *relevant operator* due to a large anomalous dimension $\gamma_m = D - 2$ at $g = g^*$, i.e., $\dim(\bar{\psi}\psi)^2 = 2(D - 1 - \gamma_m) = 2 < D$, while the would-be “counter terms” $(\bar{\psi}\psi)^4$ and $\partial_\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)$ are *irrelevant operators*, $\dim(\bar{\psi}\psi)^4 = 4(D - 1 - \gamma_m) = 4 > D$, $\dim[\partial_\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)] = 2(D - \gamma_m) = 4 > D$. At $D = 4$, however, all these operators equally have dimension $4(= D)$ and become *marginal operators*. Hence they should be included in order to make the theory renormalizable, in which case the NJL model in its renormalizable version becomes identical to the Higgs-Yukawa system (σ model, or “standard model”)³⁶.

5. Gauged Nambu-Jona-Lasinio Model

Now we discuss the gauged NJL model in four dimensions which was shown to be renormalizable due to large anomalous dimension, $1 < \gamma_m < 2$, in a sense similar to the renormalizability of $\text{NJL}_{<4}$.^{15,16,17,18,20}

5.1. Ladder Schwinger-Dyson Equation and Critical Line

Let us start with the Lagrangian of the gauged NJL model, the NJL model (4.1) plus $SU(N)$ gauge theory:

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - e\cancel{A})\psi + \frac{G}{2N} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau^a\psi)^2] - \frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}), \quad (5.1)$$

where e is the gauge coupling constant. Here we first discuss the case of non-running gauge coupling (“standing” limit of walking gauge coupling). In the ladder approximation the SD equation takes the same form as that of the QED plus NJL model. In Euclidean space, the ladder SD equation for the fermion propagator $iS^{-1}(p) = Z_\psi^{-1}(p^2)(\not{p} - \Sigma(p^2))$ in Landau gauge takes the form (after angular integration):

$$\Sigma(x) = \frac{g}{\Lambda^2} \int_0^{\Lambda^2} dy \frac{y\Sigma(y)}{y + \Sigma(y)^2} + \int_0^{\Lambda^2} dy \frac{y\Sigma(y)}{y + \Sigma(y)^2} K(x, y), \quad (5.2)$$

where $x \equiv p^2$, $K(x, y) = \lambda / \max(x, y)$ and $\lambda \equiv (3C_2(\mathbf{F})/4\pi)(e^2/(4\pi))$, with $C_2(\mathbf{F}) = (N^2 - 1)/2N$ being the quadratic Casimir of the fermion representation $\mathbf{F}(= \mathbf{N})$, respectively. (Note that $Z_\psi(p^2) \equiv 1$ in Landau gauge in the ladder approximation.) The dynamical mass function is normalized as $\Sigma(m^2) = m$ as before.

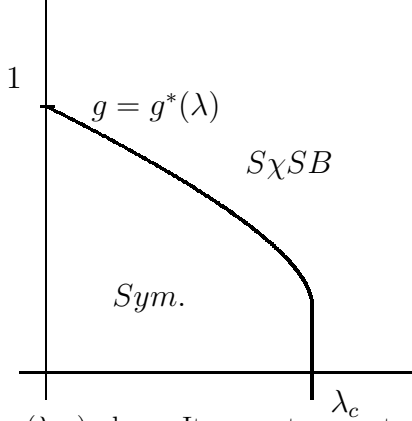


Fig.1 Critical line in (λ, g) plane. It separates spontaneously broken ($S\chi SB$) phase and unbroken phase ($Sym.$) of the chiral symmetry.

Eq.(5.2) was first studied by Bardeen, Leung and Love³⁷ in QED for the strong gauge coupling region $\lambda > \lambda_c = 1/4$. A full set of $S\chi SB$ solutions in the whole (λ, g) plane and the *critical line* were found by Kondo, Mino and Yamawaki and independently by Appelquist, Soldate, Takeuchi and Wijewardhana.¹⁴ In particular, at $0 < \lambda < \lambda_c = 1/4$ the asymptotic form of the solution of the ladder SD equation (5.2) takes the form:¹⁴

$$\Sigma(p^2)_{p \gg m} \simeq m \left(\frac{p}{m} \right)^{-1+\omega} \quad (0 < \lambda < \lambda_c), \quad (5.3)$$

which is reduced to a constant mass function $\Sigma(p^2) \equiv \text{const.} = m$ in the pure NJL limit ($\lambda \rightarrow 0$) as it should be. At $\lambda = \lambda_c$ we have $\Sigma(p^2) \sim 1/p$ ($\omega = 0$) up to logarithm (See Section 6.1).

The critical line in the (λ, g) plane is a generalization of the critical coupling in NJL model. It is the line of the second-order phase transition separating spontaneously broken ($m/\Lambda \neq 0$) and unbroken ($m/\Lambda = 0$) phases of the chiral symmetry (Fig.1)¹⁴:

$$\begin{aligned} g &= \frac{1}{4}(1+\omega)^2 \equiv g^*, & \omega &\equiv \sqrt{1 - \frac{\lambda}{\lambda_c}} & (0 < \lambda < \lambda_c), \\ \lambda &= \lambda_c = \frac{1}{4} & (g < \frac{1}{4}). \end{aligned} \quad (5.4)$$

Here the overall mass scale m at $0 < \lambda < \lambda_c$ satisfies the scaling relation similar to (4.4):¹⁴

$$\frac{2}{1-\omega^2} \cdot \left(\frac{m}{\Lambda} \right)^{2\omega} \simeq \frac{1}{g^*} - \frac{1}{g}, \quad (5.5)$$

where the factor $2/(1-\omega^2)$ in L.H.S. gives a divergence in the pure NJL limit $\omega \rightarrow 1$ ($\lambda \rightarrow 0$), which actually corresponds to the logarithmic divergence in the pure NJL

model, (4.3). (A careful analysis at $\omega \rightarrow 1$ in fact yields logarithmic divergence)³⁸. Again the absence of this logarithmic factor at $\omega \neq 1$ ($\lambda \neq 0$) is relevant to the renormalizability of the gauged NJL model.

As in the pure NJL model Eq.(5.5) indicates that the dynamical mass m sharply rises as we move away from the critical coupling $g = g^*$. Now the critical coupling on the critical line $g = g^*(\lambda)$ does depend on the value of gauge coupling λ , and vice versa $\lambda = \lambda^*(g)$. This again means that *even a tiny difference (symmetry violation) of λ (or g) for the same g (or λ) can cause amplified effects on the dynamical mass; $m = 0$ (below the critical line) or $m \neq 0$ (above the critical line).*

5.2. Large Anomalous Dimension and Renormalizability/Nontriviality

Again the scaling relation (5.5) leads to the beta function

$$\beta(g) = 2\omega g \left(1 - \frac{g}{g^*}\right) \quad (g \simeq g^*), \quad (5.6)$$

while the solution (5.3) yields the condensate

$$\langle \bar{\psi}\psi \rangle_\Lambda = -Tr S(p) = -\frac{N}{4\pi^2} \int_0^{\Lambda^2} dx \frac{x\Sigma(x)}{x + \Sigma(x)^2} \simeq \left(\frac{\Lambda}{m}\right)^{1+\omega} \cdot \langle \bar{\psi}\psi \rangle_m, \quad (5.7)$$

which implies $Z_m^{-1} \sim \Lambda^{1+\omega}$ and hence a large anomalous dimension:⁶

$$\gamma_m = 1 + \omega \quad (0 < \omega \equiv \sqrt{1 - \frac{\lambda}{\lambda_c}} < 1) \quad (5.8)$$

at the critical line. In particular, we reproduce $\gamma_m = 2$ in the pure NJL model ($\lambda = 0$) and $\gamma_m = 1$ for $\lambda = \lambda_c$. The solution of the SD equation (5.3) is also compared with the general result from OPE and RGE, (2.66):

$$\Sigma(p^2) \underset{p \gg m}{\simeq} \frac{m^3}{p^2} \left(\frac{p}{m}\right)^{\gamma_m}, \quad (5.9)$$

where we have set $\mu = m$ and $\langle \bar{\psi}\psi \rangle_\mu \simeq -m^3$. Then we find that such a slowly damping solution (5.3) actually corresponds to a large anomalous dimension (5.8).

As in NJL_{<4} model, induced kinetic term and induced quartic coupling of the auxiliary field ϕ in the gauged NJL model remain *finite* in the continuum limit $\Lambda \rightarrow \infty$, this time thanks to the power damping behavior of the mass function (5.3) in contrast to the constant mass function in the pure NJL model. Such a damping behavior is due to the presence of gauge interactions. This finiteness in turn implies the finiteness of the effective Yukawa coupling in the continuum limit, namely, the *nontriviality* of the gauged NJL model. This can also be seen from the PS formula (2.64) whose integral is now convergent thanks to the power-damping behavior of $\Sigma(p^2)$.

Actually, an explicit renormalization procedure in the ladder approximation was performed by Kondo, Tanabashi and Yamawaki¹⁶ through the effective potential as in NJL_{<4}. The fine-tuning of the bare coupling $1/g^* - 1/g \ll 1$ in (5.5) corresponds to the continuum limit $\Lambda/m \rightarrow \infty$, which now defines a finite renormalized theory explicitly written in terms of renormalized quantities, in sharp contrast to the pure NJL model where a similar fine-tuning through (4.3) or (4.4) has nothing to do with a finite renormalized theory. This renormalization leads to the beta function and the anomalous dimension:¹⁶

$$\beta(g) = 2\omega g \left(1 - \frac{g}{g^*}\right), \quad (5.10)$$

$$\gamma_m(g) = 1 - \omega + 2\omega \frac{g}{g^*} \quad (5.11)$$

at $0 < \lambda < \lambda_c$, where g is *either renormalized or bare* coupling. These expressions are valid both in the $S\chi SB$ and unbroken phases. Note that these RGE functions at $g \simeq g^*$ take the same form as (5.6) and (5.8) obtained from the scaling relation and the condensate in $S\chi SB$ phase. It is now clear that the critical line $g = g^* = \frac{1}{4}(1+\omega)^2$ is a UV fixed line where the anomalous dimension takes the large value

$$1 < \gamma_m(g = g^*) = 1 + \omega < 2. \quad (5.12)$$

The essence of the renormalizability now resides in the fact that this dynamics possesses a large anomalous dimension $\gamma_m > 1$ but not too large, $\gamma_m < 2$.¹⁶ It in fact implies that the four-fermion interactions are *relevant operators*, $2 < d_{(\bar{\psi}\psi)^2} = 2(3 - \gamma_m) = 4 - 2\omega < 4$.⁶ Accordingly, possible *higher dimensional interactions*, $(\bar{\psi}\psi)^4$, $\partial_\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)$, etc., are *irrelevant operators* ($d > 4$ due to $d_{\bar{\psi}\psi} > 1$), in contrast to the pure NJL model ($\omega = 1$) where these operators are marginal ones ($d = 4$ due to $d_{\bar{\psi}\psi} = 1$). Thus the *presence of the gauge interactions* can change drastically the four-fermion theories into *renormalizable theories* without introducing “higher dimensional operators”.^{15,16,17,18,19,20} These higher dimensional operators are in fact calculated to be finite and hence no counter terms are needed.

5.3. Running Effects of Gauge Coupling

One can easily take account of perturbative *running effects* of the $SU(N)$ gauge coupling, typically the QCD coupling, in the ladder SD equation (“improved ladder SD equation”)³⁹ by simply replacing λ in (5.2) by the one-loop running one $\lambda(p^2)$ parameterized as follows:

$$\lambda(p^2) = \begin{cases} \lambda_\mu & (p < \mu_{IR}) \\ \frac{A}{4 \ln(p/\Lambda_{QCD})} & (p > \mu_{IR}) \end{cases}, \quad (5.13)$$

where $A = c/b = 18C_2(\mathbf{F})/(11N - 2N_f)$ ($= 24/(33 - 2N_f)$ for $N = 3$) and $\lambda_\mu (= (A/4)/\ln(\mu_{IR}/\Lambda_{QCD}))$ are constants and $\mu_{IR}(= O(\Lambda_{QCD}))$ an artificial “IR cutoff” of

otherwise divergent running coupling constant (We choose $\lambda_\mu > 1/4$ so as to trigger the $S_\chi SB$ already in the pure QCD).

Then the SD equation takes the form

$$\Sigma(x) = \frac{g}{\Lambda^2} \int_0^{\Lambda^2} dy \frac{y \Sigma(y)}{y + \Sigma^2(y)} + \int_0^{\Lambda^2} dy \frac{y \Sigma(y)}{y + \Sigma^2(y)} \mathbf{K}(x, y), \quad (5.14)$$

where $\mathbf{K}(x, y) \equiv \lambda(\max(x, y, \mu_{IR}^2)) / \max(x, y)$. Note that the non-running case is regarded as the “standing” limit $A \rightarrow \infty$ (with $\lambda_\Lambda \equiv \lambda(\Lambda^2)$ fixed) of the walking coupling ($A \gg 1$).⁴²

The $S_\chi SB$ solution of (5.14) is logarithmically damping⁶,

$$\Sigma(p^2) \simeq m \cdot \left(\frac{\ln \frac{p}{\Lambda_{QCD}}}{\ln \frac{m}{\Lambda_{QCD}}} \right)^{-\frac{A}{2}}, \quad (5.15)$$

which is essentially the same as (5.3) with a small power $\lambda(\sim \lambda_\Lambda) \ll 1$. In the case of pure QCD ($g = 0$), such a very slowly damping solution (“irregular asymptotics”) is the *explicit* chiral-symmetry-breaking solution due to the current quark mass ($\gamma_m \simeq 2\lambda(p^2) = A/(2\bar{t}) \ll 1$, see (2.56)).^{28,39} However, Miransky and Yamawaki⁶ pointed out that it can be the *$S_\chi SB$ solution in the presence of an additional four-fermion interaction*. Accordingly, the condensate is quadratically enhanced as in pure NJL model except for a logarithmic correction:

$$\langle \bar{\psi} \psi \rangle_\Lambda = -\frac{N}{4\pi^2} \int_0^{\Lambda^2} dx \frac{x \Sigma(x)}{x + \Sigma(x)^2} \simeq \left(\frac{\Lambda}{m} \right)^2 \left(\frac{\ln \frac{\Lambda}{\Lambda_{QCD}}}{\ln \frac{m}{\Lambda_{QCD}}} \right)^{-\frac{A}{2}} \cdot \langle \bar{\psi} \psi \rangle_m. \quad (5.16)$$

The solution (5.15) corresponds to a very large anomalous dimension

$$\gamma_m \simeq 2 - 2\lambda(p^2), \quad (5.17)$$

or

$$\exp \left[\int_0^t \gamma_m(t') dt' \right] = \exp \left[2t - \frac{A}{2} \ln \left(\frac{\bar{t}}{\bar{t}_\mu} \right) \right] = \left(\frac{p}{m} \right)^2 \left(\frac{\ln \frac{p}{\Lambda_{QCD}}}{\ln \frac{m}{\Lambda_{QCD}}} \right)^{-\frac{A}{2}}, \quad (5.18)$$

(compare with (2.57)). For $p^2 = \Lambda^2$ we have $\gamma_m(g^*) = 2 - 2\lambda_\Lambda$ near the “critical line”

$$g = g^* \simeq 1 - 2\lambda_\Lambda \quad (5.19)$$

at $\lambda_\Lambda \ll 1$.^{6,41,15,32} (There is no critical line in the rigorous sense in this case, since $S_\chi SB$ takes place in the whole coupling region due to pure QCD dynamics which yields dynamical mass $m = m_{QCD} = O(\Lambda_{QCD})$.) Note that (5.17) and (5.19) coincide with those in the non-running case, (5.8) and (5.4), respectively, at $\lambda \ll 1$.

As to the renormalizability of the gauged NJL model in this case, convergence of the kinetic term and the quartic self-coupling of ϕ is the same as the convergence of f_π through the PS formula (2.64), which depend on the power of the logarithmic damping factor in (5.15), $\sim \ln p^{-A/2}$. Thus $A > 1$ is the condition for the convergence of f_π and hence for the renormalizability.^{15,16,17,18,19,20} This point will be explained further in the Section 7.

6. Tightly Bound Composite Higgs Models

Now that we have seen that the gauged NJL model is an explicit dynamics which has a large anomalous dimension and sensible continuum limit, we can discuss possible applications of it to the electroweak symmetry breaking, namely the tightly bound composite Higgs models mentioned before. There are a variety of tightly bound composite Higgs models based on the gauged NJL model; walking TC ($\gamma_m \simeq 1$)^{4,5}, strong ETC TC ($1 < \gamma_m < 2$)⁶ and top quark condensate ($\gamma_m \simeq 2$)^{8,9,10,11,12}, etc., whose order parameters are all enhanced by the factor

$$\exp \left[\int_0^t \gamma_m(t') dt' \right], \quad (6.1)$$

where t could be $\ln(p/m)$ or $t_\Lambda (= \ln(\Lambda/m))$.

6.1. Walking Technicolor

It was first pointed out by Yamawaki, Bando and Matumoto⁵ that the technicolor within the *ladder SD equation* (with the gauge coupling constant *fixed*, i.e., non-running) possesses an *S χ SB solution* with a large anomalous dimension:

$$\gamma_m \simeq 1, \quad (6.2)$$

$$\Sigma(p^2) \sim \frac{1}{p} \quad (p \gg \Lambda_{TC}), \quad (6.3)$$

$$\langle \bar{\psi}\psi \rangle_{\Lambda_{ETC}} \simeq \left(\frac{\Lambda_{ETC}}{\Lambda_{TC}} \right) \cdot \langle \bar{\psi}\psi \rangle_{\Lambda_{TC}}, \quad (6.4)$$

and hence resolves the long standing problems of the old TC, a scale-up of QCD, in a sense discussed in Section 3. Essentially the same observation was also made by Akiba and Yanagida⁵ without notion of anomalous dimension. It should also be mentioned that Holdom⁴ earlier recognized the same dynamics through a purely *numerical* analysis of the ladder-type SD equation (without notion of anomalous dimension).

The above feature is actually the essence of the “walking TC”, a generic name currently used (see Appelquist et al.⁵) for a wider class of TC’s with $\gamma_m \simeq 1$ due to slowly running (“walking”, $A \gg 1$) gauge coupling *including* the non-running ($A \rightarrow \infty$, “standing”) case as an extreme case. In order for the walking TC to be a

realistic solution of the FCNC problem, however, it must be very close to the standing limit anyway.⁴² In the standing limit the $S\chi SB$ solution exists only when the gauge coupling λ exceeds a critical value $\lambda_c = 1/4$. Hence the critical coupling plays a role of a nontrivial UV fixed point. Thus the walking TC may be viewed as the TC with a nontrivial UV fixed point/pseudo fixed point, with the coupling being kept close to the critical coupling all the way up to Λ_{ETC} scale.

Moreover, as we mentioned before, $\gamma_m = 1$ is realized in the gauged NJL model at $\lambda = \lambda_c$. Actually, it was suggested¹⁵ that the non-zero four-fermion coupling $g \simeq 1/4$ at $\lambda = \lambda_c$ might be “*induced*” by the dynamics of the (standing) gauge theory itself. If it is the case, the standing/walking TC might be realized at $(\lambda, g) = (\lambda_c, 1/4)$ but not at $(\lambda_c, 0)$ as was considered originally.

6.2. Strong ETC Technicolor

Next we come to the TC with even larger anomalous dimension, $1 < \gamma_m < 2$, which is due to strong four-fermion coupling $g \simeq g^*$ arising from ETC interaction, Pati-Salam interaction, preonic interaction, etc.. This is generically dubbed a strong ETC technicolor⁶. Based on the $S\chi SB$ solution¹⁴, we have an even bigger enhancement of the order parameters due to such a large anomalous dimension:

$$1 < \gamma_m = 1 + \sqrt{1 - \frac{\lambda}{\lambda_c}} < 2, \quad (6.5)$$

$$\Sigma(p^2) \sim p^{-1 + \sqrt{1 - \frac{\lambda}{\lambda_c}}}, \quad (6.6)$$

$$\langle \bar{\psi}\psi \rangle_{\Lambda_{ETC}} \simeq \left(\frac{\Lambda_{ETC}}{\Lambda_{TC}} \right)^{1 + \sqrt{1 - \frac{\lambda}{\lambda_c}}} \cdot \langle \bar{\psi}\psi \rangle_{\Lambda_{TC}}. \quad (6.7)$$

This *in principle* can yield enhancement of the quark mass (3.4), say, up to $O(\Lambda_{TC})$ and could account for a large top quark mass (if one takes $\gamma_m \simeq 2$).

6.3. Towards the Top Quark Condensate

Once we have taken a TC with such an extremely tightly bound composite Higgs with $\gamma_m \simeq 2$ in order to accommodate a large top quark mass, we may consider a much simpler alternative: Namely, the top quark itself may play the role of the technifermion triggering the electroweak symmetry breaking. In fact, a top quark condensate was proposed by MTY^{8,9}, based on the $S\chi SB$ solution of the SD equation for the gauged NJL model (this time QCD plus four-fermion interactions) with $\lambda_{QCD} \ll 1$ (see (5.15)-(5.17)):

$$\gamma_m \simeq 1 + \sqrt{1 - \frac{\lambda_{QCD}}{\lambda_c}} \simeq 2 - \frac{\lambda_{QCD}}{2\lambda_c} \simeq 2 - \frac{A}{2t}, \quad (6.8)$$

$$\Sigma(p^2) \sim \left(\ln \frac{p}{\Lambda_{QCD}} \right)^{-\frac{A}{2}}, \quad (6.9)$$

$$\langle \bar{t}t \rangle_\Lambda \simeq \left(\frac{\Lambda}{m_t} \right)^2 \left(\frac{\ln \frac{\Lambda}{\Lambda_{QCD}}}{\ln \frac{m_t}{\Lambda_{QCD}}} \right)^{-\frac{A}{2}} \cdot \langle \bar{t}t \rangle_{m_t}, \quad (6.10)$$

where μ was taken as the top quark mass m_t . This model will be explained in somewhat details in the next section.

7. Top Quark Condensate

7.1. Why Top Quark Condensate?

Recently the elusive top quark has been finally discovered and found to have a mass of about 180 GeV,²⁴ roughly on the order of weak scale 250 GeV. This is extremely large compared with mass of all other quarks and leptons and seems to suggest a special role of the top quark in the electroweak symmetry breaking, *the origin of mass*, and hence a strong connection with the Higgs boson itself.

Such a situation can be most naturally understood by the top quark condensate proposed by Miransky, Tanabashi and Yamawaki (MTY)^{8,9} and by Nambu¹⁰ independently. This entirely replaces the standard Higgs doublet by a composite one formed by a strongly coupled short range dynamics (four-fermion interaction) which triggers the top quark condensate. The Higgs boson emerges as a $\bar{t}t$ bound state and hence is deeply connected with the top quark itself. Thus the model may be called “top mode standard model”⁹ in contrast to the SM (may be called “Higgs mode standard model”). The model was further developed by the renormalization-group (RG) method.^{11,12}

Once we understand that the top quark mass is of the weak scale order, then the question is why other quarks and leptons have very small mass compared with the weak scale. Actually, the Yukawa coupling is dimensionless and hence naturally expected to be of $O(1)$. This is the question that MTY^{8,9} had solved in the top quark condensate through the amplification of the symmetry violation in the critical phenomenon.

MTY⁸ introduced explicit four-fermion interactions responsible for the top quark condensate in addition to the standard gauge couplings. Based on the explicit solution of the ladder SD equation^{14,6}, MTY found that even if all the dimensionless four-fermion couplings are of $O(1)$, only the coupling larger than the critical coupling yields non-zero (large) mass, while others do just zero masses. This is a salient feature of the *critical phenomenon* or the *dynamics with large anomalous dimension* as was already explained in NJL and gauged NJL models. Combined with the PS formula²⁹, MTY predicted the top quark mass as a *decreasing* function of the cutoff Λ and in particular the minimum value to be about 250 GeV for the Planck scale

cutoff, which actually coincides with the weak scale.*

The model was further formulated in an elegant fashion by Bardeen, Hill and Lindner (BHL)¹² in the SM language, based on the RG equation and the compositeness condition. BHL essentially incorporates $1/N_c$ sub-leading effects such as those of the composite Higgs loops and $SU(2)_L \times U(1)_Y$ gauge boson loops which were disregarded by the MTY formulation. We can explicitly see that BHL is in fact *equivalent* to MTY at $1/N_c$ -leading order. Such effects turned out to reduce the above MTY value 250 GeV down to 220 GeV, a somewhat smaller value but still on the order of the weak scale.

Although the prediction appears to be substantially higher than the experimental value mentioned above²⁴, there still remains a possibility that (at least) an essential feature of the top quark condensate idea may eventually survive. As a possible modification within the simplest version of the model we shall experiment with an idea to take the cutoff beyond the Planck scale. Even if we were allowed to ignore the quantum gravity effects, however, we cannot take the cutoff beyond the Landau pole of $U(1)_Y$ gauge coupling, which actually yields an absolute minimum value of the top mass prediction $m_t \simeq 200\text{GeV}$.

On the other hand, if the standard gauge groups are unified into a (“walking”)* GUT, we may take the cutoff to infinity thanks to the renormalizability^{15,16,17,18,19,20} of the gauged NJL model with “walking” ($A > 1$) gauge coupling. We shall consider this possibility (“top mode walking GUT”)²³ in which the top and Higgs mass prediction is controlled by the Pendleton-Ross (PR) infrared fixed point⁴³ at GUT scale and can naturally lead to $m_t \simeq m_H \simeq 180\text{GeV}$.

7.2. The Model

Let us first explain the original version of the top quark condensate model (top mode standard model) proposed by MTY^{8,9} based on explicit four-fermion interactions. The model consists of the standard three families of quarks and leptons with the standard $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge interactions but *without Higgs doublet*. Instead of the standard Higgs sector MTY introduced $SU(3)_C \times SU(2)_L \times U(1)_Y$ -invariant *four-fermion interactions among quarks and leptons*, the origin of which is expected to be a new physics not specified at this moment. The new physics determines the ultraviolet (UV) scale (cutoff Λ) of the model, in contrast to the infrared (IR) scale (weak scale $F_\pi \simeq 250\text{GeV}$) determined by the mass of W/Z bosons.

*It should be emphasized that the MTY prediction (receipt date: Jan. 3, 1989)⁸ was made when the lower bound of the top quark mass through direct experiment was only 28 GeV (TRISTAN value) and many theorists (including SUSY enthusiasts) were still expecting the value below 100 GeV. It in fact appeared absurd at that time to claim a top mass on the order of weak scale. Thus such a large top mass was really a *prediction* of the model.

*In the context of renormalizability of the gauged NJL model, we shall use in this section “walking” for $A = c/b > 1$ (slowly running) instead of the usual definition of walking $A \gg 1$ (*very* slowly running including non-running case).

The explicit form of such four-fermion interactions reads: ^{8,9}

$$\begin{aligned}\mathcal{L}_{4f} = & \left[G^{(1)}(\bar{\psi}_L^i \psi_R^j)(\bar{\psi}_R^j \psi_L^i) \right. \\ & + G^{(2)}(\bar{\psi}_L^i \psi_R^j)(i\tau_2)^{ik}(i\tau_2)^{jl}(\bar{\psi}_L^k \psi_R^l) \\ & \left. + G^{(3)}(\bar{\psi}_L^i \psi_R^j)(\tau_3)^{jk}(\bar{\psi}_R^k \psi_L^i) \right] + \text{h.c.},\end{aligned}\quad (7.1)$$

where i, j, k, l are the weak isospin indices and $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$ are the four-fermion coupling constants among top and bottom quarks $\psi \equiv (t, b)$. It is straightforward^{8,9} to include other families and leptons into this form.

The symmetry structure (besides $SU(3)_C$) of the four-fermion interactions, $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$, is $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$, $SU(2)_L \times SU(2)_R \times U(1)_V$ and $SU(2)_L \times U(1)_Y \times U(1)_V \times U(1)_A$, respectively. The $G^{(2)}$ term is vital to the mass of the bottom quark in this model.^{8,9} In the absence of the $G^{(2)}$ -term, (7.1) possesses a $U(1)_A$ symmetry which is explicitly broken only by the color anomaly and plays the role of the Peccei-Quinn symmetry.⁹

Let us disregard the $G^{(2)}$ term for the moment, in which case the MTY Lagrangian (7.1) simply reads

$$\mathcal{L}_{4f} = G_t(\bar{\psi}_L t_R)^2 + G_b(\bar{\psi}_L b_R)^2 + \text{h.c.}, \quad (7.2)$$

with $G_t \equiv G^{(1)} + G^{(3)}$ and $G_b \equiv G^{(1)} - G^{(3)}$. This MTY Lagrangian with $G_b = 0$ was the starting point of BHL¹², but setting $G_b = 0$ overlooks an important aspect of the top quark condensate, as we will see in the following.

7.3. Why $m_t \gg m_{b,c,\dots}$?

We now explain one of the key points of the model, i.e., explicit dynamics which gives rise to a large isospin violation in the condensate $\langle \bar{t}t \rangle \gg \langle \bar{b}b \rangle$ ($m_t \gg m_b$), or more generally, naturally explains why only the top quark has a very large mass. MTY found^{8,9} that critical phenomenon, or theory having nontrivial UV fixed point with large anomalous dimension, is actually such a dynamics, based on the $S_\chi SB$ solution of the ladder SD equation for the gauged NJL model. Such an amplification of symmetry violation was already explained in NJL model and gauged NJL model in the previous sections.

For the $SU(3)_c \times SU(2)_L \times U(1)_Y$ -gauged NJL model, the ladder SD equation becomes simpler in the large N_c limit: Rainbow diagrams of the $SU(2)_L \times U(1)_Y$ gauge boson lines are suppressed compared with those of the QCD gluon lines. Thus we consider the ladder SD equation with QCD coupling and four-fermion coupling (7.2). Without $G^{(2)}$ term, the top and bottom quarks satisfy decoupled SD equations, each equation being the same form as (5.14) with different four-fermion couplings $g_t \neq g_b$ ($g^{(3)} \neq 0$). (We can easily find a solution for the SD equation with the $G^{(2)}$ term.)⁹ For simplicity, we first consider the *non-running* QCD coupling, in which case (5.14) is reduced to (5.2). Then there exists a critical line (5.4) around which

the dynamical mass m in (5.5) sharply rises from zero to order $O(\Lambda)$ as we move away from the critical coupling $g = g^*(\lambda)$. In view of the critical line and the critical behavior (5.5), MTY^{8,9} indeed found *amplified isospin symmetry violation* for a small (*however small*) violation in the coupling constants. Thus we have an S χ SB solution with *maximal isospin violation*, $m_t \neq 0$ and $m_b = 0$, when

$$g_t > g^* = \frac{1}{4}(1 + \omega)^2 > g_b \quad (7.3)$$

(g_t is above the critical line and g_b is below it). As already mentioned, we need *not* to set $G_b = 0$ in the four-fermion interactions (7.2) to obtain $m_b = 0$. Thus, even if we assume that all the dimensionless couplings are $O(1)$, the *critical phenomenon* naturally explains why only the top quark can have a large mass, or more properly, *why other fermions can have very small masses*: $m_t \gg m_{b,c,\dots}$. It is indeed realized if only the top quark coupling is above the critical coupling, while all others below it: $g_t > g^* > g_{b,c,\dots} \implies m_t \neq 0, m_{b,c,\dots} = 0$. Note that other couplings do *not* need to be zero nor *very* small.

In the case of running gauge coupling (5.14), we have already seen that essentially the same critical phenomenon takes place through the presence of the “critical line” (5.19): We again have an S χ SB solution with *maximal isospin violation*, $m_t \neq 0, m_b = 0$ (apart from $m_{QCD} \ll m_t$), under a condition similar to (7.3); $g_t > g^*(\simeq 1 - 2\lambda_\Lambda) > g_b$.

7.4. Top Quark Mass Prediction

Now we come to the central part of the model, namely, relating the dynamical mass of the condensed fermion (top quark) to the mass of W/Z bosons.

The top quark condensate $\langle \bar{t}t \rangle$ indeed yields a standard gauge symmetry breaking pattern $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$ to feed the mass of W and Z bosons. Actually, the mass of W and Z bosons in the top quark condensate is generated via dynamical Higgs mechanism as in the technicolor, (3.3), where $F_\pi (\simeq 250\text{GeV})$ determine the IR scale of the model, this time the top quark mass.

7.4.1. SD Equation plus PS Formula (MTY)

The decay constants of these composite NG bosons F_π may be calculated through the formula (2.61) which are written in terms of the fermion mass function and the amputated BS amplitude. The BS amplitude is a solution of the BS equation which must be solved consistently with the SD equation for the fermion propagator.⁴⁴ Instead, here we use the PS formula (2.64)²⁹ for simplicity, which was generalized by MTY⁸ to the $SU(2)$ -asymmetric case, $m_t \neq m_b$ and $m_{t,b} \neq 0$:

$$F_{\pi^\pm}^2 = \frac{N_c}{8\pi^2} \int_0^{\Lambda^2} dx x \cdot$$

$$\frac{(\Sigma_t^2 + \Sigma_b^2) - \frac{x}{4}(\Sigma_t^2 + \Sigma_b^2)' + \frac{x}{2}(\Sigma_t^2 - \Sigma_b^2) \left[\frac{1 + (\Sigma_t^2)'}{x + \Sigma_t^2} - \frac{1 + (\Sigma_b^2)'}{x + \Sigma_b^2} \right]}{(x + \Sigma_t^2)(x + \Sigma_b^2)}, \quad (7.4)$$

$$F_{\pi^0}^2 = \frac{N_c}{8\pi^2} \int_0^{\Lambda^2} dx x \cdot \left[\frac{\Sigma_t^2 - \frac{x}{4}(\Sigma_t^2)'}{(x + \Sigma_t^2)^2} + \frac{\Sigma_b^2 - \frac{x}{4}(\Sigma_b^2)'}{(x + \Sigma_b^2)^2} \right]. \quad (7.5)$$

Let us consider the extreme case, the maximal isospin violation mentioned above, $\Sigma_t(p^2) \neq 0$ and $\Sigma_b(p^2) = 0$. We further take a “toy” case switching off the gauge interactions: $\Sigma_t(p^2) \equiv \text{const.} = m_t$ (pure NJL limit). Then (7.4) and (7.5) are both *logarithmically divergent* at $\Lambda/m_t \rightarrow \infty$ with the same coefficient:

$$F_{\pi^\pm}^2 = \frac{N_c}{8\pi^2} m_t^2 \left[\ln \frac{\Lambda^2}{m_t^2} + \frac{1}{2} \right], \quad (7.6)$$

$$F_{\pi^0}^2 = \frac{N_c}{8\pi^2} m_t^2 \ln \frac{\Lambda^2}{m_t^2}. \quad (7.7)$$

Now, we could predict m_t by fixing $F_{\pi^\pm} \simeq 250\text{GeV}$ so as to have a correct m_W through (3.3). Actually, (7.6) determines m_t as a *decreasing function of cutoff* Λ . The largest physically sensible Λ (new physics scale) would be the Planck scale $\Lambda \simeq 10^{19}\text{GeV}$ at which we have a minimum value prediction $m_t \simeq 145\text{GeV}$. If we take the limit $\Lambda \rightarrow \infty$, we would have $m_t \rightarrow 0$, which is nothing but triviality (no interaction) of the pure NJL model: $y_t \equiv \sqrt{2}m_t/F_\pi \rightarrow 0$ at $\Lambda \rightarrow \infty$.

One might naively expect a disastrous weak isospin violation for the maximal isospin-violating dynamical mass, $m_t \neq 0$ and $m_b = 0$. However, for $\Lambda \gg m_t$, (7.6) and (7.7) yield $F_{\pi^\pm} \simeq F_{\pi^0}$, or

$$\delta\rho \equiv \frac{F_{\pi^\pm}^2 - F_{\pi^0}^2}{F_{\pi^\pm}^2} = \frac{N_c m_t^2}{16\pi^2 F_{\pi^\pm}^2} \simeq \frac{1}{2 \ln \frac{\Lambda^2}{m_t^2}} \ll 1. \quad (7.8)$$

Then the problem of weak isospin relation can in principle be solved *without custodial symmetry*. Actually, the isospin violation $F_{\pi^\pm} \neq F_{\pi^0}$ in (7.4) and (7.5) solely comes from the different propagators having different $\Sigma(p^2)$, essentially the IR quantity, which becomes less important for $\Lambda \gg m$, since the integral is UV dominant. This is the essence of the “dynamical mechanism” of MTY to save the isospin relation $\rho \simeq 1$ without custodial symmetry.

Now in the gauged NJL model, QCD plus four-fermion interaction (7.2), essentially the same mechanism as the above is operative. Based on the very slowly damping solution of the ladder SD equation (5.15) and the PS formulas (7.4)-(7.5), MTY^{8,9} predicted m_t and $\delta\rho$ as the *decreasing function of cutoff* Λ . For the Planck scale cutoff $\Lambda \simeq 10^{19}\text{GeV}$, we have^{8,9*}

$$m_t \simeq 250\text{GeV}, \quad (7.9)$$

*One may substitute into (5.15) the *numerical* solution (instead of the analytical one (5.15)) of the ladder SD equation (5.14), the result being the same as (7.9).⁴⁵

$$\delta\rho \simeq 0.02 \ll 1. \quad (7.10)$$

This is compared with the pure NJL case $m_t \simeq 145\text{GeV}$: The QCD corrections are quantitatively rather significant (Presence of the gauge coupling also changes the qualitative feature of the theory from a nonrenormalizable/trivial theory into a renormalizable/nontrivial one.)^{15,16,17,18,19,20}

It will be more convenient to write an *analytical* expression for F_π . Neglecting the derivative terms with $\Sigma_t(x)'$ and using (5.15), we may approximate (7.4) as¹⁷

$$\begin{aligned} F_\pi^2 &\simeq \frac{N_c}{8\pi^2} \int_{m_t^2}^{\Lambda^2} dx \frac{\Sigma_t^2}{x} \\ &\simeq \frac{N_c m_t^2}{16\pi^2} \frac{A}{A-1} \frac{(\lambda(m_t^2))^{A-1} - (\lambda(\Lambda^2))^{A-1}}{(\lambda(m_t^2))^A}. \end{aligned} \quad (7.11)$$

This analytic expression was first obtained by Marciano¹¹ in the case of $A = 8/7$ ($N_f = 6$), which actually reproduces the MTY prediction (7.9).

7.4.2. RG Equation plus Compositeness Condition (BHL)

Now, we explain the BHL formulation¹² of the top quark condensate, which is based on the RG equation combined with the compositeness condition. BHL start with the SM Lagrangian which includes explicit Higgs field at the Lagrangian level:

$$\mathcal{L}_{SM} = -y_t(\bar{\psi}_L^i t_R \phi_i + \text{h.c.}) + (D_\mu \phi^\dagger)(D^\mu \phi) - m_H^2 \phi^\dagger \phi - \lambda_4 (\phi^\dagger \phi)^2, \quad (7.12)$$

where y_t and λ_4 are Yukawa coupling of the top quark and quartic interaction of the Higgs, respectively. BHL imposed “compositeness condition” on y_t and λ_4 in such a way that (7.12) becomes the *MTY Lagrangian* (7.2) (with $G_b = 0$):

$$\frac{1}{y_t^2} \rightarrow 0, \quad \frac{\lambda_4}{y_t^4} \rightarrow 0 \quad \text{as } \mu \rightarrow \Lambda, \quad (7.13)$$

where μ is the renormalization point above which the composite dynamics are integrated out to yield an effective theory (7.12). Thus the compositeness condition implies divergence at $\mu = \Lambda$ of both the Yukawa coupling of the top quark and the quartic interaction of the Higgs.

Now, in the one-loop RG equation, the beta function of y_t is given by

$$\beta(y_t) = \frac{y_t^3}{(4\pi)^2} \left(N_c + \frac{3}{2} \right) - \frac{y_t}{(4\pi)^2} \left(3 \frac{N_c^2 - 1}{N_c} g_3^2 + \frac{9}{4} g_2^2 - \frac{17}{12} g_1^2 \right), \quad (7.14)$$

where g_1 , g_2 and g_3 are the gauge couplings of $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$, respectively. BHL solved the RG equation for the beta function (7.14) combined with the compositeness condition (7.13) as a boundary condition at $\mu = \Lambda$.

7.4.3. BHL versus MTY

Let us first demonstrate¹⁷ that *in the large N_c limit* BHL formulation¹² is *equivalent* to that of MTY^{8,9}, both based on the same MTY Lagrangian (7.2). In the $N_c \rightarrow \infty$ limit for (7.14), we may neglect the factor $3/2$ in the first term (composite Higgs loop effects) and g_2^2 and g_1^2 in the second term (electroweak gauge boson loops), which corresponds to the similar neglect of $1/N_c$ sub-leading effects in the ladder SD equation in the MTY approach. Then (7.14) becomes simply:

$$\frac{dy_t}{d\mu} = \beta(y_t) = N_c \frac{y_t^3}{(4\pi)^2} - \frac{3N_c y_t g_3^2}{(4\pi)^2}. \quad (7.15)$$

Within the same approximation the beta function of the QCD gauge coupling reads

$$\frac{dg_3}{d\mu} = \beta(g_3) = -\frac{1}{A} \frac{3N_c g_3^3}{(4\pi)^2}. \quad (7.16)$$

Solving (7.15) and (7.16) by imposing the compositeness condition at $\mu = \Lambda$, we arrive at¹⁷

$$y_t^2(\mu) = \frac{2(4\pi)^2}{N_c} \frac{A-1}{A} \frac{(\lambda(\mu^2))^A}{(\lambda(\mu^2))^{A-1} - (\lambda(\Lambda^2))^{A-1}}. \quad (7.17)$$

Noting the usual relation $m_t^2 = \frac{1}{2} y_t^2(m_t) v^2$ ($v = F_\pi$), we obtain

$$\frac{m_t^2}{F_\pi^2} = \frac{y_t^2(m_t)}{2} = \frac{(4\pi)^2}{N_c} \frac{A-1}{A} \frac{(\lambda(m_t^2))^A}{(\lambda(m_t^2))^{A-1} - (\lambda(\Lambda^2))^{A-1}}. \quad (7.18)$$

This is precisely the same formula as (7.11) obtained in the MTY approach based on the SD equation and the PS formula.* Thus we have established

$$\text{BHL}(\frac{1}{N_c} \text{ leading}) = \text{MTY}. \quad (7.19)$$

Having established equivalence between MTY and BHL in the large N_c limit, we now comment on the relation between them in more details. Note that MTY formulation is based on the nonperturbative picture, ladder SD equation and PS formula, which is valid at $1/N_c$ leading order, or the NJL bubble sum with ladder-type QCD corrections (essentially the leading log summation). MTY extrapolated this $1/N_c$ leading picture all the way down to the low energy region where the sub-leading effects may become important.

*Alternatively, we may define $F_\pi^2(\mu^2) \equiv 2m_t^2/y_t^2(\mu)$ which coincides with the integral (7.11) with the IR end m_t^2 simply replaced by μ^2 . Then the compositeness condition (7.13) reads $F_\pi^2(\mu^2 = \Lambda^2) = 0$ (no kinetic term of the Higgs).

On the other hand, BHL is crucially based on the perturbative picture, one-loop RG equation, which can easily accommodate $1/N_c$ sub-leading effects in (7.14) such as the loop effects of composite Higgs and electroweak gauge bosons. However, BHL formalism must necessarily be combined with the compositeness condition (7.13). The compositeness condition is obviously inconsistent with the perturbation and is a purely nonperturbative concept based on the same $1/N_c$ leading NJL bubble sum as in the MTY formalism. Thus the BHL perturbative picture breaks down at high energy near the compositeness scale Λ where the couplings y_t and λ_4 blow up as required by the compositeness condition.

So there must be a certain matching scale $\Lambda_{\text{Matching}}$ such that the perturbative picture (BHL) is valid for $\mu < \Lambda_{\text{Matching}}$, while only the nonperturbative picture (MTY) becomes consistent for $\mu > \Lambda_{\text{Matching}}$.^{*} Such a point may be defined by the energy region where the two-loop contributions dominate over the one-loop ones. However, thanks to the presence of a *quasi-infrared fixed point*⁴⁷, BHL prediction is *numerically* quite stable against ambiguity at high energy region, namely, rather independent of whether this high energy region is replaced by MTY or something else. Then we expect $m_t \simeq m_t(\text{BHL}) = \frac{1}{\sqrt{2}}y_t(\mu = m_t)v \simeq \frac{1}{\sqrt{2}}\bar{y}_t v$ within 1-2 %, where \bar{y}_t is the quasi-infrared fixed point given by $\beta(\bar{y}_t) = 0$ in (7.14). The composite Higgs loop changes \bar{y}_t^2 by roughly the factor $N_c/(N_c + 3/2) = 2/3$ compared with the MTY value, i.e., $250\text{GeV} \rightarrow 250 \times \sqrt{2/3} = 204\text{GeV}$, while the electroweak gauge boson loop with opposite sign pulls it back a little bit to a higher value. The BHL value¹² is then given by

$$m_t = 218 \pm 3\text{GeV}, \quad \text{at } \Lambda \simeq 10^{19}\text{GeV}. \quad (7.20)$$

The Higgs boson was predicted as a $t\bar{t}$ bound state with a mass $M_H \simeq 2m_t$ ^{8,9,10} based on the pure NJL model calculation¹. Its mass was also calculated by BHL¹² through the full RG equation of λ_4 , the result being

$$M_H = 239 \pm 3\text{GeV} \quad \left(\frac{M_H}{m_t} \simeq 1.1\right) \quad \text{at } \Lambda \simeq 10^{19}\text{GeV}. \quad (7.21)$$

If we take only the $1/N_c$ leading terms, we would have the mass ratio $M_H/m_t \simeq \sqrt{2}$, which was also obtained through the ladder SD equation.⁴⁸

7.5. Top Mode Walking GUT

As we have seen, the top quark condensate naturally explains, through the critical phenomenon, why only the top quark mass is much larger than that of other quarks

^{*}Of course, the $1/N_c$ leading picture might be subject to ambiguity such as the possible higher dimensional operators, cutoff procedures, etc., all related to the nonrenormalizability of the NJL model.³⁶ These problems will be conceptually solved and phenomenologically tamed, when coupled to the (“walking” ($A > 1$)) gauge interactions (renormalizability of the gauged NJL model)^{15,16,17,18,19,20} to be discussed later. Here we just comment that even if there might be such an ambiguity, the $1/N_c$ picture (MTY) is the only consistent way to realize the compositeness condition as was done by the BHL paper itself.

and leptons: $m_t \gg m_{b,c,\dots}$. It further predicts the top mass on the order of weak scale. However, the predicted mass 220GeV is somewhat larger than the mass of the recently discovered top quark, $176\text{GeV} \pm 13\text{GeV}$ (CDF) and $199 + 38 / - 36\text{GeV}$ (D0)²⁴. Here we shall discuss a possible remedy of this problem within the simplest model based on the MTY Lagrangian (7.2).²³

7.5.1. Landau Pole Scenario

First we recall that the top mass prediction is a *decreasing* function of the cutoff Λ . Then the simplest way to reduce the top mass would be to raise the cutoff as much as possible. Let us assume that quantum gravity effects would not change drastically the physics described by the low energy theory without gravity. Then we may raise the cutoff Λ beyond the Planck scale up to the Landau pole $\Lambda \simeq 10^{41}\text{GeV}$ where the $U(1)_Y$ gauge coupling g_1 diverges and the SM description itself stops to be self-consistent. In such a case the top and Higgs mass prediction becomes:

$$m_t \simeq 200\text{GeV}, \quad M_H \simeq 209\text{GeV} \quad \text{at } \Lambda \simeq 10^{41}\text{GeV} \quad (7.22)$$

which is the absolute minimum value of the prediction within the simplest version of the top quark condensate.

If it is really the case, it would imply composite $U(1)_Y$ gauge boson and composite Higgs generated *at once by the same dynamics*, since the Landau pole then may be regarded as a BHL compositeness condition also for the vector bound state as well as the composite Higgs. Actually, we can formulate the BHL compositeness condition for vector-type four-fermion interactions (Thirring-type four-fermion theory) as a *necessary condition* for the formation of a vector bound state. The possibility that both the Higgs and $U(1)_Y$ gauge boson can be composites by the same dynamics may be illustrated by an explicit model, the Thirring model in $D(2 < D < 4)$ dimensions. Reformulated as a gauge theory through hidden local symmetry²⁶, the Thirring model was shown to have the dynamical mass generation, which implies that a composite Higgs and a composite gauge boson are generated at the same time.⁴⁹

At any rate, the prediction of this scenario $m_t \simeq 200\text{GeV}$ still seems to be a little bit higher than the experimental value, although the situation is not very conclusive yet. Then we shall consider another possibility, namely, taking the cutoff to infinity: $\Lambda \rightarrow \infty$. In order to do this we should first recall the previous discussions on the renormalizability of the gauged NJL model with “walking” gauge coupling ($A > 1$).^{15,16,17,18,19,20}

7.5.2. More on Renormalizability of Gauged NJL Model

This phenomenon was first pointed out by Kondo, Shuto and Yamawaki¹⁵ through the convergence of F_π in the PS formula for the solution of the SD equation (5.15) in the four-fermion theory plus QCD. Contrary to the logarithmic divergence of (7.6) in

the pure NJL model, it was emphasized that for $A > 1$ we have a *convergent* integral for F_π and hence a nontrivial (interacting) theory with finite effective Yukawa coupling $y_t \equiv \sqrt{2}m_t/F_\pi \neq 0$ in the continuum limit: Namely, the presence of “walking” ($A > 1$) gauge interaction changes the trivial/nonrenormalizable theory (pure NJL model) into a nontrivial/renormalizable theory (gauged NJL model).¹⁵

The analytical expression of the effective Yukawa coupling is already given by (7.11) (MTY), which is equivalent to (7.18) obtained as a solution of the RG equation with a compositeness condition at $1/N_c$ leading (BHL). From this expression it was again noted¹⁷ that *iff* $A > 1$ (“walking” gauge coupling with $N_c \sim N_f \gg 1$), then the effective Yukawa coupling remains finite, $y_t > 0$, in the continuum limit $\Lambda \rightarrow \infty$. This is in sharp contrast to the triviality of the pure NJL model in which $y_t \rightarrow 0$ in the continuum limit as was mentioned earlier.

It was further pointed out by Kondo, Tanabashi and Yamawaki¹⁶ that this renormalizability is equivalent to existence of a PR infrared fixed point⁴³ for the gauged Yukawa model. The PR fixed point is given by the solution of $\frac{d(y_t/g_3)}{d\mu} = 0$ with (7.15) and (7.16):

$$y_t^2 = \frac{(4\pi)^2}{N_c} \frac{A-1}{A} \lambda, \quad (7.23)$$

where $\lambda = 3C_2(\mathbf{F})g_3^2/(4\pi)^2$. Similar argument was recently developed more systematically by Harada, Kikukawa, Kugo and Nakano.²⁰

As to the non-running (standing) case ($A \rightarrow \infty$), the integral for F_π^2 is more rapidly convergent, since $\Sigma(p^2)$ is power damping, (5.3), instead of logarithmic damping. In this case the renormalization procedure was performed explicitly by Kondo, Tanabashi and Yamawaki¹⁶ through the effective potential in the ladder approximation as was already explained in Section 5.

7.5.3. Top Mode Walking GUT

In view of the renormalizability of the gauged NJL model with “walking” gauge coupling, we may take the $\Lambda \rightarrow \infty$ limit of the top quark condensate. However, in the realistic case we actually have the $U(1)_Y$ gauge coupling which, as it stands, grows at high energy to blow up at Landau pole and hence invalidates the above arguments of the renormalizability. Thus, in order to apply the above arguments to the top quark condensate, we must remove the $U(1)_Y$ gauge interaction in such a way as to unify it into a GUT with “walking” coupling ($A > 1$) beyond GUT scale. Then the renormalizability requires that the GUT coupling at GUT scale should be determined by the PR infrared fixed point.²³

For a simple-minded GUT with $SU(N)$ group, the PR fixed point takes the form similar to (7.23):

$$y_t^2(\Lambda_{\text{GUT}}) = \frac{3C_2(\mathbf{F})}{N} \frac{A-1}{A} g_{\text{GUT}}^2(\Lambda_{\text{GUT}}) \simeq \frac{3}{2} g_{\text{GUT}}^2(\Lambda_{\text{GUT}}),$$

$$\lambda_4(\Lambda_{\text{GUT}}) = \frac{6C_2(\mathbf{F})}{N} \frac{(A-1)^2}{A(2A-1)} g_{\text{GUT}}^2(\Lambda_{\text{GUT}}) \simeq \frac{3}{2} g_{\text{GUT}}^2(\Lambda_{\text{GUT}}), \quad (7.24)$$

where we assumed $N \gg 1$ and $A \gg 1$ ($N_f \sim N \gg 1$) for simplicity. Then the top Yukawa coupling at GUT scale is essentially determined by the GUT coupling at GUT scale up to some numerical factor depending on the GUT group and the representations of particle contents. Using “effective GUT coupling” including such possible numerical factors, we may perform the BHL full RG equation analysis for $\mu < \Lambda_{\text{GUT}} \simeq 10^{15} \text{ GeV}$ with the boundary condition of the above PR fixed point at GUT scale.

For typical values of the effective GUT coupling $\alpha_{\text{GUT}} \equiv g_{\text{GUT}}^2/4\pi = 1/40, 1/50$ and $1/60$, prediction of the top and Higgs masses reads:

$$(m_t, M_H) \simeq (189, 193), (183, 183), (177, 173) \quad \text{GeV}, \quad (7.25)$$

respectively. Note that these PR fixed point values at GUT scale are somewhat smaller than the coupling values at GUT scale which focus on the quasi-infrared fixed point in the low energy region. Thus the prediction is a little bit away from the quasi-infrared fixed point. This would be the simplest extension of the top quark condensate consistent with the recent experiment on the top quark mass.

8. Conclusion

We have discussed a variety of tightly bound composite Higgs models, walking technicolor, strong ETC technicolor and top quark condensate, based on the gauged NJL model as the explicit dynamics with large anomalous dimension and fixed point. Universal feature of this type of dynamics is the amplification of the symmetry violation due to the large anomalous dimension or the fine-tuning of the coupling near the critical point (UV fixed point). An extreme case $\gamma_m \simeq 2$ yields maximal symmetry violation, which was used to predict an exceptionally large mass of the top quark, even if the top coupling is on the same order as those of other quarks and leptons. The fine-tuning of the gauged NJL model corresponds to the renormalization which leads to the existence of the renormalizable and nontrivial continuum theory in contrast to the pure NJL model. Although the situation about top quark mass is still not yet conclusive, we hope that at least essence of the idea of the top quark condensate may eventually survive in the sense that the *origin of mass* is deeply related to the top quark mass.

Acknowledgements

We would like to thank Iwana Inukai and Masaharu Tanabashi for collaboration and discussions on the recent results presented in this lecture. This work was supported in part by the Sumitomo Foundation and a Grant-in Aid for Scientific Research from the Ministry of Education, Science and Culture (No. 05640339).

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